NUMERICAL SOLUTION OF SOME UNCERTAIN DIFFUSION PROBLEMS

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by

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“A person who never made a mistake never tried anything new".

Albert Einstein

Dedicated to

My Beloved

Parents

Department of Mathematics

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DECLARATION

I hereby declare that the work which is being presented in the thesis entitled "**Numerical solution of some uncertain diffusion problems**" for the award of the degree of Doctor of Philosophy in Mathematics, submitted in the Department of Mathematics, National Institute of Technology Rourkela, Rourkela 769008, Odisha, India, is an authentic record of my own work carried out under the supervision of Prof. (Dr.) S. Chakraverty.

The matter embodied in this thesis has not been submitted by me for the award of any other degree.

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CERTIFICATE

This is to certify that the thesis entitled ***Numerical solution of some uncertain diffusion problems*** submitted by **SUKANTA NAYAK** to National Institute of Technology Rourkela for the award of the degree of **Doctor of Philosophy** in *MATHEMATICS* is an authentic record of research work carried out by him under my guidance and supervision.

To the best of my knowledge the work incorporated in this thesis has not been submitted to any other University or Institute for the award of a degree.

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Abstract

Diffusion is an important phenomenon in various fields of science and engineering. These problems depend on various parameters viz. diffusion coefficients, geometry, material properties, initial and boundary conditions etc. Governing differential equations with deterministic parameters have been well studied. But, in real practice these parameters may not be crisp (exact) rather it involves vague, imprecise and incomplete information about the system variables and parameters. Uncertainties occur due to error in measurements, observations, experiments, applying different operating conditions or it may be due to maintenance induced errors, etc. As such, it is an important concern to model these type of uncertainties. Traditionally uncertain problems are modelled through probabilistic approach. But probabilistic methods may not able to deliver reliable results at the required precision without sufficient data. In this context, interval and fuzzy theory may be used to manage such uncertainties. Accordingly, the system parameters and variables are represented here as interval and fuzzy numbers.

Generally, we get interval or fuzzy system of equations for uncertain steady state problems with interval or fuzzy parameters whereas interval or fuzzy eigenvalue problems may be obtained for unsteady state. This thesis redefined interval or fuzzy arithmetic in order to handle the uncertain problems. The proposed arithmetic has been used to solve fuzzy and interval system of equations and eigenvalue problems. Various numerical methods viz. Finite Element Method (FEM), Wavelet Method (WM), Euler Maruyama and Milstein Methods are studied by introducing interval or fuzzy theory. The proposed arithmetic has been combined with FEM and WM to develop Interval or Fuzzy Finite Element Method (I/FFEM) and Interval or Fuzzy Wavelet Method (I/FWM). Further, it may be pointed out that sometimes systems may possess uncertainties due to randomness and fuzziness of the parameters. As such, here we have hybridized the concept of fuzziness as well as stochasticity to develop numerical fuzzy stochastic methods viz. interval or Fuzzy Euler Maruyama and Interval/Fuzzy Milstein. These methods are also been used to solve various diffusion problems.

Numerical examples and different application problems are solved to demonstrate the efficiency and capabilities of the developed methods. In this respect, imprecisely defined diffusion problems such as heat conduction and conjugate heat transfer in rod, homogeneous and non-homogeneous fin and plate, along with one group, multi group and point kinetic neutron diffusion with interval or fuzzy uncertainties have been investigated. The convergence of the field variables have been investigated with respect to the number of element discretization of the domain in case of I/FEM. Accordingly, convergence of the proposed interval or fuzzy FEM has been studied for unsteady heat conduction in a cylindrical rod. For conjugate heat transfer problems, the convergence of uncertain temperature distributions with respect to the number of element discretizations has also been studied. Further, various combinations of uncertain parameters are considered and the sensitivity of these parameters has been reported. Next, one group and two group problems have been solved and the sensitivity of the uncertain parameters in the context of fast and thermal neutrons are presented. The hybrid fuzzy stochastic methods have also been used to investigate uncertain stochastic point kinetic neutron diffusion problem. Uncertain variation of neutron populations are analysed by considering two random samples. Developed interval or fuzzy WM has also been used to solve uncertain differential equation. Finally obtained results for the said problems are compared in special cases for the validation of proposed methods.

Keywords: Interval, fuzzy set, fuzzy number, α-cut, diffusion problems, interval or fuzzy system of equations, interval or fuzzy eigenvalue problem, interval or fuzzy finite element method, interval or fuzzy wavelet method, heat conduction, conjugate heat transfer, one group neutron diffusion, multi group neutron diffusion, Fuzzy Euler Maruyama Method (FEMM), Fuzzy Milstein Method (FMM).

Chapter 1

Introduction

Chapter 1

Introduction

Diffusion is an important phenomenon in various fields of science and engineering. It may arise in a variety of problems viz. heat transfer, fluid flow and neutron diffusion etc. Corresponding problems may be modelled by different types of differential equations. The type of differential equation depends upon the problem at hand, the parameters, coefficients involved and on other operating conditions. In real practice, the parameters used in the modelled physical problem are not crisp (or exact) because of the experimental error, mechanical defect, measurement error etc. In that case the problem has to be defined with uncertain parameters which make it challenging to investigate.

Basically diffusion problem such as heat transfer leads to a system of simultaneous equations by using different numerical methods. Similarly the propagation problem turns into eigenvalue problem. Generally (as mentioned above) the corresponding differential equations are solved by considering the involved parameters as crisp. This simplifies the problem to a great extent and in this context corresponding solution methods are already available in literature. Moreover, the diffusion problems become more complex when we consider non-homogeneous cases or anisotropic medium. The above complexities in the model make the problem interesting even for crisp parameters.

Although, the uncertainties are handled by various authors using probability density functions or statistical methods. But these methods need plenty of data and also may not consider the vague or imprecise parameters. Accordingly, one may use interval and/or fuzzy computation in the analysis of the problems. In this investigation, most of the uncertain diffusion problems have been solved by using Finite Element Method (FEM) which may be called as Interval or Fuzzy Finite Element Method (I/FFEM). Systems may sometimes possess uncertainties due to both randomness and fuzziness of the parameters too. As such, we have hybridized the concept of fuzziness and stochasticity to develop numerical fuzzy stochastic techniques viz. Fuzzy Euler Maruyama and Fuzzy Milstein methods. These methods have also been used here to solve few diffusion problems. Another computationally efficient technique viz. Wavelet Method (WM) has also been used along with the interval/fuzzy uncertainty.

As said above that interval and fuzzy computations are used recently as a tool to handle the vagueness of parameters. In this respect, finite element method has been used here when the uncertain parameters are in term of interval and/or fuzzy. Accordingly, new computation method with interval and fuzzy values has been developed for reducing the computational effort. As mentioned earlier, applying interval/fuzzy finite element, we get either interval/fuzzy system of equations or eigenvalue problems depending upon the uncertain diffusion problems. Few authors proposed different methods for the solution of interval or fuzzy valued system of equations and eigenvalue problems. But, sometimes those are not efficient rather problem dependent. Those methods also fail sometimes when fully interval or fuzzy systems are considered. As such, the target of the present investigation is to develop new methods to handle various uncertain diffusion problems. In the following paragraph, first a literature review is included to have a handy knowledge of the present problem(s) till date.

1.1 Literature Review

As mentioned above, diffusion problems convert either into system of simultaneous equations or to eigenvalue problems. The solutions for interval/fuzzy system of linear equations are studied by various researchers. Few authors also discussed the method of uncertain bound of eigenvalues. (Sevastjanov & Dymova 2009) investigated a new method for solving both the interval/fuzzy equations for linear case. (Friedman et al. 1998) used the embedding approach to solve *n* by *n* fuzzy linear system of equations. Some authors (Abbasbandy & Alvi 2005; Allahviranloo, Kermani, et al. 2008; Senthilkumar & Rajendran 2011; Li et al. 2010) proposed methods which makes more easier for finding the uncertain solutions of fuzzy system of linear equations. They have considered the coefficient matrix as crisp. Also some authors (Allahviranloo, Kermani, et al. 2008; Liu 2010; Nasseri & Zahmatkesh 2010) have taken the coefficient matrix as fuzzy. (Allahviranloo, Mikaeilvand, et al. 2008) have taken all positive values for the coefficient matrix and used parametric form of linear system. Whereas, (Nasseri & Zahmatkesh 2010) used Huang method for computing a nonnegative solution of the fully fuzzy linear system of equations and (Liu, 2010) developed an approximate method to solve fully fuzzy linear system of equations. On the other hand, some authors discussed fuzzy eigenvalue problem in (Gersem et al. 2005; Chiao et al. 1995; Chen & Rao 1997; Chiao 1998; Lallemand et al. 1999).

As such we discuss below various diffusion problems with respect to crisp and uncertain (interval/fuzzy) parameters. Heat transfer problems are first surveyed in the following subsection.

1.1.1 Heat Transfer Problems

Heat transfer is a common phenomenon which may be found in various fields of science and engineering. Heat transfer is actually a multi-dimensional conjugate problem, in which heat conduction takes place not only in the direction orthogonal to the walls (transverse conduction), but also parallel to them (longitudinal conduction). Conjugate heat transfer refers to a heat transfer process involving an interaction of conduction within a solid body and the convection from the solid surface to fluid moving over the surface. Therefore, a realistic analysis of conjugate heat transfer problems necessitates the coupling of the conduction in the solid and the convection in the fluid. In view of this, we may characterize the conjugate heat transfer in a plate as (i) when both plate and the surrounded fluid are at rest, (ii) when the plate is moving and surrounded fluid is at rest, (iii) when the surrounded fluid is moving and plate is at rest, and finally (iv) when both the surrounded fluid and plate are moving.

In this context, (Wijeysundera 1986) analysed a steady conjugate problem with convective boundary conditions for pipes and rectangular channels heated in a finite region, by considering the wall conduction in the axial direction. Further (Bilir & Ates 2003) investigated transient conjugate heat transfer for laminar flow in the thermal entrance region of pipes considering two dimensional wall and axial fluid conduction. Whereas, (Jahangeer et al. 2007) solved conjugate heat transfer problem of rectangular fuel element of a nuclear reactor dissipating heat into an upward moving stream of liquid sodium. (Ciofalo 2007) reviewed the influence of Longitudinal Heat Conduction (LHC) on heat exchanger performance.

Further, we know that heat spontaneously flows from a body having higher temperature to lower temperature. But in absence of external driving fluxes it approaches to thermal equilibrium. There are two types of conduction such as steady and unsteady state. Steady state conduction is a form of conduction where the temperature differences deriving by the conduction remains constant and it is independent of time. The steady state heat conduction problem is well known and its solution by exact method has been solved earlier (Carlslaw & Jaeger 1986). The analysis may be difficult when heat is transfered through a complicated domain.

Modelling of heat transfer problems may be represented by different types of differential equations. The governing differential equations although are solved earlier by various authors using exact methods (Carlslaw & Jaeger 1986; Liu et al. 1986; Bondarev 1997; Monte 2000). Using exact methods, the analysis may be difficult when heat is transfered through a complicated domain. So, various numerical techniques are proposed for these types of problems viz. finite difference, finite volume and finite element methods (Magnus & Achi 2011; Muhieddine et al. 2009). (Magnus & Achi 2011) used finite difference method in their paper to model and solve the governing ground water flow rates, flow direction and hydraulic heads through an aquifer. (Muhieddine et al. 2009) described one dimensional phase change problem. They have used vertex centered finite volume method to solve the problem. (Edward & Robert 1966) used FEM to solve heat conduction problem. A non-iterative, finite element-based inverse method for estimating surface heat flux histories on thermally conducting bodies is developed by (Ling et al. 2003). They considered both linear and non-linear problems, and sequentially minimizes the least square error norm between corresponding sets of measured and computed temperatures. Further (Onate et al. 2006) used Galerkin FEM for convective–diffusive problems with sharp gradients using finite calculus. (Basak et al. 2011) used a penalty finite element method based simulation to analyse the influence of various thermal boundary conditions of walls on mixed convection for a square cavity filled with porous medium. It may be mentioned that there are many papers on FEM related to these types of problems. But few of them are cited here for the sake of completeness. In view of the above literatures, it reveals that the traditional finite element method may easily be used where the parameters or the values are exact that is in crisp form.

But in actual practice these problems may involve uncertainties. These uncertainties may occur due to various experimental errors viz. heat transfer coefficients, heat convection coefficients, input heat rate and ambient temperatures etc. The above parameters play important role in the system characteristics. In order to handle these uncertain parameters, several probabilistic methods have been introduced by different authors. As such Monte Carlo method is generally used to solve heat and mass propagation problems. It essentially involves a large number of process samples which are obtained by numerically solving the problem for artificially generated random parameter samples. As such, Monte Carlo method has been used to analyse thermal food processes with variable parameters (Wang et al. 1991; Varga et al. 2000; Caro-Corrales et al. 2002; Demir et al. 2003; Halder et al. 2007; Laguerre & Flick 2010). (Deng & Liu 2002) implemented Monte Carlo method to solve the direct bio heat transfer problems. They have demonstrated the bio heat transfer problem with transient or space-dependent boundary conditions, blood perfusion, metabolic rate, and volumetric heat source for tissue. (Wu 2009) developed a Monte Carlo method to simulate transient radiative transfer in a refractive planar medium exposed to a collimated pulse irradiation. Further, (Kovtanyuk & Nikolai 2012) have considered radiative–conductive heat transfer in a medium bounded by two reflecting and radiating plane surfaces. The author proposed a recursive algorithm based on some modification of the Monte Carlo method and utilized the diffusion approximation of the radiative transfer equation.

In particular, it is very difficult to get a large number of experimental data so we need an alternative method in which we may handle the uncertainty considering few experimental data. In this context (Zadeh 1965) proposed an alternate idea that is fuzzy theory to handle uncertainty. The direct implementation of interval or fuzzy becomes more complex and the computation sometimes become difficult task. So, to avoid such difficulty various authors tried different techniques. As such, (Dong & Shah 1987) proposed vertex method for computing functions of fuzzy variables and (Dong & Wong 1987) used Fuzzy Weighted Average Method (FWAM) for fuzzy comutation. (Yang et al. 1993) discussed the calculation of functions with fuzzy numbers. They developed methods which require less computation than the FWAM. Then (Klir 1997) revised fuzzy arithmetic by considering the relevant requisite constraints. Further, (Hanss 2002) gave a transformation method based on the concept of α -cut where the fuzzy arithmetic is reduced to interval computation.

As metioned above, heat transfer problem leads to a system of simultaneous equations by using FEM and presence of uncertainty makes the system of simultaneous equations uncertain. Accordingly, (Matinfar et al. 2008) used householder decomposition method to solve fuzzy linear equations and they considered only the right hand side column vector as fuzzy and solved some example problems. For the fuzzy coefficient matrix $\tilde{A}$, (Panahi et al. 2008) obtained lower triangular and upper triangular matrix separately. (Senthilkumar & Rajendran 2011) considered symmetric coefficient matrix to solve Fuzzy Linear System (FFLS) of equations. They decomposed the coefficient matrix by using Cholesky method. However, (Vijayalakshmi & Sattanathan 2011) introduced Symmetric times Triangular (ST) decomposition procedure to solve fully fuzzy system of linear equations. (Behera & Chakraverty 2013b) proposed a method to solve fuzzy real system of linear equations by solving two $n \times n$ crisp systems of linear equations. Here the coefficient matrix is considered as real crisp, whereas an unknown variable vector and right hand side vector are considered as fuzzy. The general system is solved by adding and subtracting the left and right bounds of the vectors respectively. Further, (Behera & Chakraverty 2012; Behera & Chakraverty 2013a) solved complex fuzzy system of linear equations also. In view of the above literatures, the present work comprises a simple arithmetic is presented to handle fuzzy system of linear equations.

Next we incorporate below the related important literatures of neutron diffusion problems.

1.1.2 Neutron Diffusion Problems

The scattering neutrons produced by fission have a high range of energies. In a nuclear reactor, these neutrons are slowed down by scattering collisions with atomic nuclei until they are thermalized. In thermal energy region, the neutrons exchange energy with the moderator atoms. So there is up-scattering of neutrons such that neutrons gain energy as well as the common down-scattering occur and neutrons loose energy. As a result of various interactions, the neutron energies in a reactor core range vary approximately from 10 MeV to 0.001 eV. These energy ranges are divided into a finite number of discrete energy groups. Hence we get multigroup neutron diffusion equations.

Due to the complicacy of the system, various semi-analytical methods are used to investigate the above problem. Recently introduced semi-analytical methods such as homotopy solutions, Adomian Decomposition Method (ADM), Differential Transform Method (DTM) and Variational Interation Method (VIM) (Hajmohammadi & Nourazar 2014; Hajmohammadi et al. 2012; Yulianti et al. 2010) are used.

In this respect, (Biswas et al. 1976) have given a method of generating stiffness matrices for the solution of multi group diffusion equation by natural coordinate system. (Azekura 1980) has also proposed a new representation of finite element solution technique for neutron diffusion equations. The author has applied this technique to two types of one-group neutron diffusion equations to test its accuracy. Further, (Cavdar & Ozgener 2004) developed a finite element-boundary element hybrid method for one or two group neutron diffusion calculations. In their paper linear or bilinear finite element formulation for the reactor core and a linear boundary element technique for the reflector are combined through interface continuity conditions which constitute the basis of the developed method. (Dababneh et al.

2011) formulated an alternative analytical solution of the neutron diffusion equation for both infinite and finite cylinders of fissile material using the homotopy perturbation method. Further, (Rokrok et al. 2012) applied Element-Free Galerkin (EFG) method to solve neutron diffusion equation in X–Y geometry. It reveals from the above literature that the neutron diffusion equations are solved by using finite element method in presence of crisp parameters only.

Uncertainty plays a vital role in nuclear science engineering problems too. These uncertainties occur due to incomplete data, impreciseness, vagueness, experimental error and different operating conditions influenced by the system. Different authors proposed various methods to handle uncertainty. As mentioned earlier, they have used probabilistic or statistical method as a tool to handle the uncertain parameters. In this context, Monte Carlo method is an alternate approach which is based on the statistical simulation of the random numbers generated on the basis of a specific sampling distribution. Monte Carlo methods have been used to solve the neutron diffusion equation with variable parameters. As such, (Nagaya et al. 2010) implemented Monte Carlo method to estimate the effective delayed neutron fraction β_{eff} . Further, (Nagaya & Mori 2011) proposed a new method to estimate the effective delayed neutron fraction β_{eff} in Monte Carlo calculations. In the above paper, the eigenvalue method is jointly used with the differential operator and correlated sampling techniques, whereas, (Shi & Petrovic 2011) used Monte Carlo methods to solve one-dimensional two-group problems and then they proved its validity. (Sjenitzer & Hoogenboom 2011) gave an analytical procedure to compute the variance of the neutron flux in a simple model of a fixed-source calculation. Recently, (Yamamoto 2012) investigated the neutron leakage effect specified by buckling to generate group constants for use in reactor core designs using Monte Carlo method.

As such in the above process we need a good number of observed data or experimental results to analyse the problem. Sometimes it may not be possible to get a large number of data. As pointed out earlier, (Zadeh, 1965) proposed an alternate idea viz. fuzzy theory to handle uncertain and imprecise variables. The presence of uncertain parameters makes the system uncertain and we get uncertain governing differential equations. In this context, uncertain fuzzy parameters are considered to solve neuclear diffusion problems using finite element method and we call it as Fuzzy Finite Element Method (FFEM). (Nicolai et al. 2011) solved the uncertain solution of heat conduction problem. In this paper authors gave a good comparison between response surface method and other methods. Recently, (Chakraverty & Nayak 2012) also solved the interval/fuzzy distribution of temperature along a cylindrical rod. Accordingly, we have used interval or fuzzy parameters to take care of the uncertainties. In general traditional interval/fuzzy arithmetic are complicated to investigate the problem. Thus, we have proposed a new technique for fuzzy arithmetic to overcome such difficulty.

Next section includes probabilistic and possibilistic uncertainties along with their hybridization.

1.1.3 Stochastic Differential Equations

The concept of Stochastic Differential Equation (SDE) has been initiated by a great philosopher Einstein in 1905 (Sauer 2012). A mathematical connection between microscopic random motion of particles and the macroscopic diffusion equation has been presented. Later it has been seen that the SDE model plays a prominent role in a range of application areas such as mathematics, physics, chemistry, mechanics, biology, microelectronics, economics and finance. Earlier the SDEs were solved by using Ito integral as an exact method which is discussed in (Malinowski & Michta 2011). But using exact method it is noticed that there occur some difficulty to study nontrivial problems and hence approximation methods are used. In this context a plenty of papers are available but we have mentioned here which are directly related to this problem. In 1982, (Rumelin 1982) defined general Runge-Kutta approximations for the solution of stochastic differential equations and an explicit form of the correction term have been given. (Kloeden & Platen 1992) are discussed about the numerical solutions of stochastic differential equation. Discrete time strong and weak approximation methods has been used by (Platen 1999) to investigate the solution of stochastic differential equations. Next, (Higham 2001) gave a major contribution to solve the approximate solutions of stochastic differential equations. Further, (Higham & Kloeden, 2005) investigated nonlinear stochastic differential equations numerically. They presented two implicit methods for Ito stochastic differential equations (SDEs) with Poisson-driven jumps. The first method is a split-step extension of the backward Euler method and the second method arises from the introduction of a compensated, martingale, form of the Poisson process. (Hayes & Allen 2005) solved stochastic point kinetic reactor problem. They modelled the point stochastic reactor problem into ordinary time dependent stochastic differential equation and studied the stochastic behaviour of the neutron flux.

It may be noted from the literature review that, previous authors have investigated the stochastic differential equations which contain crisp parameters. But in general, the involved parameters may not be crisp rather these may be uncertain. Here, the uncertain parameters are considered again as Triangular Fuzzy Number (TFN) and Trapezoidal Fuzzy Number (TRFN). As such, (Kim 2005) considered fuzzy sets space for real line and the existence and uniqueness of the solution is obtained. The solution is investigated by taking particular conditions which are imposed on the structure of integrated fuzzy stochastic processes such that a maximal inequality for fuzzy stochastic Ito integral holds. Next, (Ogura 2008) proposed an approach to solve Fuzzy Stochastic Differential Equation (FSDE) which does not contain any notion of fuzzy stochastic Ito integral and the method was based on the selection of sets. Further, (Malinowski & Michta 2011) presented the existence and uniqueness of solutions to the FSDEs driven by Brownian motion and the continuous dependence on initial condition and stability properties are studied.

Finally, new concept viz. uncertainties in wavelet method are introduced. To the best of our knowledge this study may be first of its kind. Accordingly, comprehensive literature reviews for wavelet method with crisp parameters have been surveyed.

1.1.4 Wavelet Method

Wavelet method is a powerful technique to investigate various science and engineering problems. There are two types of wavelets such as discrete and continuous. In the present work, discrete type of wavelet is considered. Discrete orthogonal wavelets are family of functions with compact support which form a basis on a bounded domain. The orthogonal wavelet family may be defined by a set of L filter coefficients $\{a_l : l = 0, 1, ..., L-1\}$, where L is an even integer. Two fundamental scale equations in wavelet theory are defined as (Chen et al. 1996)

$$\phi(x) = \sum_{l=0}^{L-1} a_l \phi(2x - l)$$

$$\psi(x) = \sum_{l=0}^{L-1} (-1)^l a_{1-l} \phi(2x - l)$$

where $\phi(x)$ and $\psi(x)$ are the scaling and wavelet functions with fundamental support over finite intervals $[0, L-1]$ and $[1-L/2, L/2]$, respectively. These equations are used to determine the value of the scaling and wavelet functions at dyadic points $x = n/2^J$, $n = 0, 1, ...$ The scaling functions at resolution level J may be defined as follows

$$\phi_{J,k}(x) = 2^{J/2}\phi(2^J x - k), \quad k \in Z .$$

In this context, (Daubechies 1992) constructed a family of orthonomal bases of compactly supported wavelets for the space of square-integrable functions, $L^2(R)$. Due to the fact that they possess several useful properties, such as orthogonality, compact support, exact representation of polynomials to a certain degree, and ability to represent functions at different levels of resolution, Daubechies wavelets have gained great interest in the numerical solutions of ordinary and partial differential equations. (Beylkin 1992) described the exact and explicit expressions of differential operators in orthonormal bases of compactly supported wavelets as well as the representations of Hilbert transform which are applied to multidimensional convolution operators. Further various finite integrals whose integrands are product of Daubechies compactly supported wavelets and their derivatives are evaluated by (Chen et al. 1996). (Avudainayagam & Vani 2000) used wavelet bases to the solution of integro-differential equations and two simple nonlinear integro-differential equations are investigated.

Haar wavelet is a special type of Daubechies wavelets. Haar family of wavelet is used by (Lepik & Tamme 2004) to obtain the numerical solution of linear integral equations. The numerical solution of five different integral equations with their exact solution have been compared in that paper. Then (Lepik 2005) used Haar wavelet technique to solve ordinary and partial differential equations. Whereas (Mehra 2009) discussed some computational aspects of wavelets and various wavelet methods. Further, (Lepik 2012) analysed free and forced vibrations of cracked Euler-Bernoulli beams by using Haar wavelet method.

In the above literature review it has been seen that the problems are again studied for crisp parameters only. But in general no system is ideal and there always involves some uncertainty. As mentioned earlier, that these uncertainties occur due to incomplete data, impreciseness, vagueness, experimental error and different operating conditions influenced by the system.

Probabilistic and statistical methods essentially involve a large number of process samples which are obtained by experimentally observing the problem for artificially generated random parameter samples. So practically it may not be possible sometimes to get a large number of data because it needs more number of experiments to perform. So, instead of probabilistic or statistical method, we may use interval parameters to handle uncertainty which may require less number of data.

1.2 Gaps

It may be seen that studies have been carried out by various authors taking crisp values for the material and geometrical properties. As such the corresponding problems of system of equations or the eigenvalue problems have been studied with crisp parameters. As said above that in practical situations the parameters or variables may involve uncertainty rather than crisp or exact. It is becoming an emerging field of research where we take the parameters or variables as uncertain in term of interval and/or fuzzy. Review of literature reveals that few authors have taken the parameters as interval or fuzzy for their investigation which are sometime problem dependent. The width of the bounds for uncertain parameters plays a dominant role in the interval and fuzzy computation. So, it is important to understand how to handle the width of the interval. Upper and lower bounds have been considered by many authors for solving various problems. But this usually gives undesired weak solution.

As such, there are many gaps in the above focused problems. It is known that interval and fuzzy computations are themselves very complex to handle. It may be noted that the subtraction of identical fuzzy numbers is not zero. Again, the multiplication of a fuzzy numbers with its inverse does not give identity element. Moreover, the multiplication and division of fuzzy numbers maximize the uncertainty drastically. Considering these, one has to develop efficient algorithms very carefully to handle these situations. It is also a great challenge to develop efficient methods for solution of interval or fuzzy valued system of equations and eigenvalue problems. The above facts may be kept in the mind while investigating the uncertain problems of science and engineering in particular to the diffusion problems.

1.3 Aims and Objectives

In view of the gaps as mentioned above, the aim of the present work is to develop new methods to handle various diffusion problems. As such, this research is focused to develop alternate interval/fuzzy arithmetic and new methods for solving interval/fuzzy algebraic systems. The efficiency and powerfulness of the proposed methods are also to be studied by investigating different diffusion problems viz. heat transfer and neutron diffusion. Further, the sensitivity of the uncertain parameters should also be analyzed. In this respect, the broad objectives related to the present research are to investigate the following:

- New method(s) for solution of uncertain system of equations;
- New method(s) for uncertain eigenvalue problems;
- Uncertain diffusion equations for heat transfer problems;
- Uncertain (with interval and fuzzy) heat convection and/or radiation time dependent (and independent) problem(s);
- Uncertain (with interval and fuzzy) one group and multi-group neutron diffusion equations;
- Fuzzy/Interval stochastic differential equations;
- Fuzzy/Interval uncertain wavelet method.

1.4 Organization of the Thesis

Present work is based on the study of uncertain analysis of various diffusion problems viz. steady and unsteady heat transfer and neutron diffusion. Accordingly, this thesis comprises ten chapters to investigate new methods for interval/fuzzy algebraic equations, interval/fuzzy stochastic differential equations, interval/fuzzy wavelet method and their applications. As such, brief outlines of each chapter are given below:

Overview of this thesis has been presented in Chapter 1. Related literatures of various diffusion problems and uncertain approaches to handle the related problems are reviewed and then gaps are identified. Accordingly, the aim and objectives of this investigation have also been included here.

Chapter 2 comprises preliminaries and basic definitions related to the present research. Here, definition of fuzzy set and its properties along with the traditional interval/fuzzy arithmetic have been presented. Various numerical techniques viz. Euler Maruyama and Milstein methods for crisp parameters are included. Finite element method and wavelet theory have also been discussed here.

New interval/fuzzy arithmetic have been proposed in Chapter 3. The proposed interval/fuzzy arithmetic have been incorporated in the Finite Element Method (FEM) and then Interval/Fuzzy FEM (I/FEM) has been developed. In order to handle the combined effect of possibility and probabilistic uncertainties in system parameters, the concept of stochasticity and fuzzy theory are hybridized. Accordingly, we have proposed Fuzzy Euler Maruyama Method (FEMM) and Fuzzy Milstein Method (FMM) to handle such scenarios. Literature review reveals that Wavelet Method (WM) is one of the powerful technique to handle propagation problems. As such, we have also incorporated the concept of interval/fuzzy theory in wavelet method and consequently Interval/Fuzzy Wavelet Method (I/FWM) has been proposed.

Chapter 4 addresses steady and unsteady state heat conduction problems with uncertain parameters. Proposed Fuzzy Finite Element Method (FFEM) (of Chapter 3) has been applied to investigate uncertain steady state heat conduction problem. In steady state problem, FFEM converts the governing differential equation into interval/fuzzy system of equation and it has been solved by the proposed methods which are discussed in Chapter 3. Similarly, FFEM reduces the unsteady state problem into uncertain eigenvalue problem. Accordingly, unsteady heat conduction problem is investigated under uncertain environment. The convergence of the results by the proposed method has also been reported here.

Chapter 5 describes the conjugate heat transfer problems with imprecisely defined parameters such that initial and boundary conditions as interval/fuzzy. Here, two different problems viz. conjugate heat transfer in a tapered fin and a plate are investigated by introducing fuzziness in the model. Governing fuzzy differential equations are solved by the proposed FFEM. The convergences of uncertain temperature distributions with respect to the number of element discretizations have been studied. Obtained results are compared in special cases. Further, the sensitivity of the uncertain parameters has also been analyzed.

Chapter 6 presents one group neutron diffusion equation for square and rectangular bare reactors with imprecisely defined parameters. Here, various coefficients and constants are considered as interval/fuzzy and the problems are modelled in terms of triangular and trapezoidal fuzzy numbers. Uncertain one group neutron diffusion equation has been solved by proposed FFEM and is investigated for different discretization of the domain.

Chapter 7 includes uncertain multi group neutron diffusion equation where diffusion coefficients, neutron interaction coefficients and group fission constants are considered as interval/fuzzy. Here, a general model for uncertain multi group neutron diffusion problem has been developed. The uncertain distributions of fast and thermal neutron populations are discussed. Considering different combinations of fuzzy parameters, various cases are investigated. A two group bench mark problem has been solved and the sensitivity of the uncertain parameters in the context of fast and thermal neutrons are presented.

In chapter 8, the concepts of fuzziness and stochasticity have been hybridized. The proposed Fuzzy Euler Maruyama and Fuzzy Milstein methods are demonstrated by considering bench mark example problems viz. Black-Scholes and Langevin stochastic differential equations. Further these methods have also been used to investigate uncertain stochastic point kinetic neutron diffusion problem. Here the uncertain variations of neutron populations are analyzed by considering two random samples. Obtained results in special case of stochastic are compared with known results.

Fuzzy theory has been combined with wavelet method in Chapter 9. In order to demonstrate the proposed method, ordinary differential equation with interval/fuzzy coefficient is considered. The proposed interval/fuzzy arithmetic (of Chapter 3) have been incorporated in wavelet theory and interval/fuzzy wavelet method is proposed. Then the proposed methods have been applied to an ordinary differential equation with interval/fuzzy coefficients. Obtained uncertain results are also compared with known results in special cases.

Chapter 10 gives the major findings and concluding remarks of the present work. Finally suggestions for future work are also included here.

Chapter 2

Preliminaries

Chapter 2

Preliminaries

A classical or crisp set A can be defined as a collection of objects or elements of universal set X. The elements of the set (say A) may be defined by using characteristic function χ_A which is

$$\chi_A : X \to \{0, 1\}, \text{ where } X \text{ is the universal set.}$$

χ_A indicates the membership of the element $x \in X$ if $\chi_A(x) = 1$ and non-membership if $\chi_A(x) = 0$.

2.1 Definitions

In the following, some important definitions related to fuzzy sets are introduced and explained.

Fuzzy set

If X is a collection of objects or elements (denoted by x) then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\} \tag{2.1}$$

where $\mu_{\tilde{A}}(x)$ is the membership function of x.

Example

Let us consider a fuzzy set $\tilde{A}$= real numbers larger than 15 (Zimmermann 1991) as $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$

where

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \le 15 \\ \left(1 + \dfrac{1}{(x-15)^2}\right)^{-1}, & x > 15 \end{cases} \tag{2.2}$$

Support of a fuzzy set

The support of a fuzzy set $\tilde{A}$ is the crisp set of elements $x \in X$ that has nonzero membership grades in $\tilde{A}$.

The support of a fuzzy set $\tilde{A}$ may be written as

$$\text{supp}(\tilde{A}) = \{x \in X \mid \mu_{\tilde{A}}(x) > 0\} \tag{2.3}$$

α-level set of a fuzzy set

The α-level set A_α of a fuzzy set $\tilde{A}$ is the crisp set of all elements $x \in X$ that belongs to the fuzzy set $\tilde{A}$ at least to the degree $\alpha \in [0,1]$.

$$A_\alpha = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha\} \tag{2.4}$$

The α-level set $A_{\alpha+}$ with

$$A_{\alpha+} = \{x \in X \mid \mu_{\tilde{A}}(x) > \alpha\} \tag{2.5}$$

is called strong α-level set of the fuzzy set $\tilde{A}$.

Convexity of a fuzzy set

A fuzzy set $\tilde{A}$ is convex if

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)), \quad x_1, x_2 \in X, \lambda \in [0,1] \tag{2.6}$$

In other words, a fuzzy set is a convex if all α-level sets are convex.

Height of a fuzzy set

The height, $h(\tilde{A})$, of a fuzzy set $\tilde{A}$ is the largest membership grade obtained by any element in that set.

$$h(\tilde{A}) = \sup_{x \in X} \tilde{A}(x) \tag{2.7}$$

A fuzzy set $\tilde{A}$ is called normal when $h(\tilde{A}) = 1$ and subnormal if $h(\tilde{A}) < 1$.

Fuzzy numbers

A fuzzy set $\tilde{A}$ is called a fuzzy number if it satisfies the following conditions:

(1) $\tilde{A}$ is normal, that is $h(\tilde{A}) = 1$;

(2) $\tilde{A}$ is convex;

(3) The membership function $\mu_{\tilde{A}}(x)$ is at least piecewise continuous.

Triangular Fuzzy Number (TFN)

A fuzzy number $\tilde{A} = [a^L, a^N, a^R]$ (Figure 2.1) is said to be triangular fuzzy number when the membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \le a^L; \\ \dfrac{x - a^L}{a^N - a^L}, & a^L \le x \le a^N; \\ \dfrac{a^R - x}{a^R - a^N}, & a^N \le x \le a^R; \\ 0, & x \ge a^R. \end{cases}$$

The Triangular Fuzzy Number (TFN) $\tilde{A} = [a^L, a^N, a^R]$ may be expressed into an ordered pair function by using $\alpha -$cut as below.

$$[\underline{f}(\alpha), \overline{f}(\alpha)] = [a^L + (a^N - a^L)\alpha,\ a^R - (a^R - a^N)\alpha], \quad \alpha \in [0, 1]$$

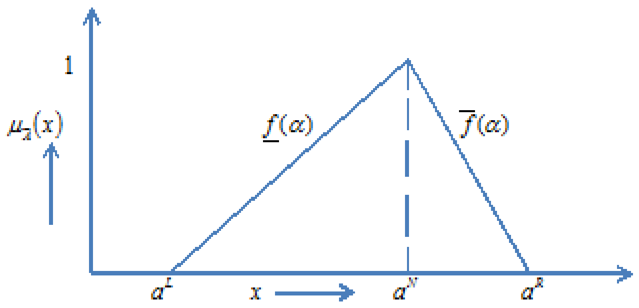

Figure 2.1. Triangular Fuzzy Number (TFN)

Trapezoidal Fuzzy Number (TRFN)

A fuzzy number $\tilde{A} = [a^L, a^{NL}, a^{NR}, a^R]$ (Figure 2.2) is said to be trapezoidal fuzzy number when the membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \le a^L; \\ \dfrac{x - a^L}{a^{NL} - a^L}, & a^L \le x \le a^{NL}; \\ 1, & a^{NL} \le x \le a^{NR}; \\ \dfrac{a^R - x}{a^R - a^{NR}}, & a^{NR} \le x \le a^R; \\ 0, & x \ge a^R. \end{cases}$$

Again, the trapezoidal fuzzy number may be expressed into an ordered pair function through $\alpha -$cut in the following manner.

$$[\underline{g}(\alpha), \overline{g}(\alpha)] = [a^L + (a^{NL} - a^L),\ a^R - (a^R - a^{NR})], \quad \alpha \in [0, 1]$$

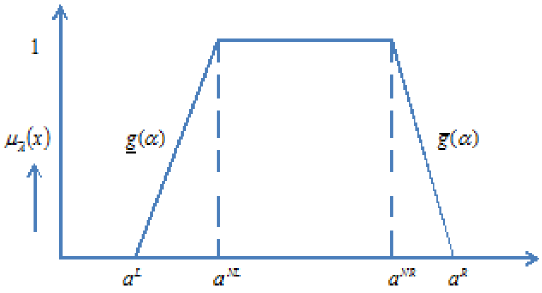

Figure 2.2. Trapezoidal Fuzzy Number (TRFN)

Interval Arithmetic

Let us consider an interval $[\underline{x}, \overline{x}]$, and then the same may be written as

$$[\underline{x}, \overline{x}] = \{x : x \in \Re, \underline{x} \le x \le \overline{x}\}$$

where $\underline{x}$ is the left value and $\overline{x}$ is the right value of the interval respectively. Let $m = \dfrac{\underline{x} + \overline{x}}{2}$ is the mid value and $w = \overline{x} - \underline{x}$ is the width of the interval $[\underline{x}, \overline{x}]$.

If $[\underline{x}, \overline{x}]$ and $[\underline{y}, \overline{y}]$ be two intervals then interval arithmetic (Neumaier 1990; Moore et al. 2014) may be written as

(1) $[\underline{x}, \overline{x}] + [\underline{y}, \overline{y}] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$ (2.8)

(2) $[\underline{x}, \overline{x}] - [\underline{y}, \overline{y}] = [\underline{x} - \overline{y}, \overline{x} - \underline{y}]$ (2.9)

(3) $[\underline{x}, \overline{x}] \times [\underline{y}, \overline{y}] = [\min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}, \max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}]$ (2.10)

(4) $[\underline{x}, \overline{x}] \div [\underline{y}, \overline{y}] = [\min\{\underline{x} \div \underline{y}, \underline{x} \div \overline{y}, \overline{x} \div \underline{y}, \overline{x} \div \overline{y}\},$

$\max\{\underline{x} \div \underline{y}, \underline{x} \div \overline{y}, \overline{x} \div \underline{y}, \overline{x} \div \overline{y}\}]$ (2.11)

Fuzzy Arithmetic

Let us consider $[\underline{x}(\alpha), \overline{x}(\alpha)]$ and $[\underline{y}(\alpha), \overline{y}(\alpha)]$ be two fuzzy numbers then fuzzy arithmetic (Zimmermann 1991; Hanss 2005) may be written as

(1) $[\underline{x}(\alpha), \overline{x}(\alpha)] + [\underline{y}(\alpha), \overline{y}(\alpha)] = [\underline{x}(\alpha) + \underline{y}(\alpha), \overline{x}(\alpha) + \overline{y}(\alpha)]$ (2.12)

(2) $[\underline{x}(\alpha), \overline{x}(\alpha)] - [\underline{y}(\alpha), \overline{y}(\alpha)] = [\underline{x}(\alpha) - \overline{y}(\alpha), \overline{x}(\alpha) - \underline{y}(\alpha)]$ (2.13)

(3) $[\underline{x}(\alpha), \overline{x}(\alpha)] \times [\underline{y}(\alpha), \overline{y}(\alpha)] =$

$[\min\{\underline{x}(\alpha)\underline{y}(\alpha), \underline{x}(\alpha)\overline{y}(\alpha), \overline{x}(\alpha)\underline{y}(\alpha), \overline{x}(\alpha)\overline{y}(\alpha)\},$ (2.14)

$\max\{\underline{x}(\alpha)\underline{y}(\alpha), \underline{x}(\alpha)\overline{y}(\alpha), \overline{x}(\alpha)\underline{y}(\alpha), \overline{x}(\alpha)\overline{y}(\alpha)\}]$

(4) $[\underline{x}(\alpha), \overline{x}(\alpha)] \div [\underline{y}(\alpha), \overline{y}(\alpha)] =$

$$[\min\{ \underline{x}(\alpha)/\underline{y}(\alpha), \underline{x}(\alpha)/\overline{y}(\alpha), \overline{x}(\alpha)/\underline{y}(\alpha), \overline{x}(\alpha)/\overline{y}(\alpha) \}, \qquad (2.15)$$

$$\max\{ \underline{x}(\alpha)/\underline{y}(\alpha), \underline{x}(\alpha)/\overline{y}(\alpha), \overline{x}(\alpha)/\underline{y}(\alpha), \overline{x}(\alpha)/\overline{y}(\alpha) \}]$$

From the above discussion we may say that the intervals are the special cases of fuzzy numbers. For each membership value (α) we get interval. As regards, if we consider a triangular fuzzy number $\tilde{A} = [a^L, a^N, a^R]$, then the same may be represented by using α-cut as below:

$$[\underline{f}(\alpha), \overline{f}(\alpha)] = [a^L + (a^N - a^L)\alpha,\ a^R - (a^R - a^N)\alpha], \quad \alpha \in [0,1].$$

If we take $\alpha = 0$, then we have the interval $[a^L, a^R]$ and at $\alpha = 1$, the interval $[a^N, a^N]$ becomes crisp.

2.2 Finite Element Method (FEM)

Finite Element Method (FEM) is a numerical technique which is used to give approximate solutions for partial differential equations. It is a tool to transform partial differential equations into algebraic equations, which are then easily solved.

The steps (Gerald & Wheatley 2013) involved in finite element method for crisp parameters are

1. Finding the functional that corresponds to the partial differential equation. This is well known for a large class of problems. The functional may be possible to develop by using Galarkin technique for weighted residue methods.
2. Subdivide the region into subregions or elements. The elements must span the entire region and approximate the boundary relatively closely.
3. Writing an interpolating relation that gives values for the dependent variable within an element based on the values at the nodes.
4. Substituting the interpolating relation into the functional, and set the partial derivatives of the functional with respect to each scalar coefficient to zero.
5. Combining together the element equations of step 4 to get a set of system equations. Adjust these for the boundary conditions of the problem, then solve it.

In subsequent chapters above preliminaries with respect to fuzzy and interval have been used. Moreover, this investigation also include probabilistic approach. Next, we discuss below few methods related to stochastic uncertainty.

2.3 Euler-Maruyama and Milstein Method

Numerical methods for the stochastic differential equations are well known. Accordingly we have considered Euler-Maruyama Method which is used to solve the said uncertain problems. As such let us assign a grid of points, $c = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = d$ and approximate x values $w_0 < w_1 < w_2 < ... < w_n$ to be determined at the respective *t* points.

Let us consider SDE initial value problem (Black & Scholes 1973)

$$\begin{cases} dX(t) = a(t, X)dt + b(t, X)dW_t \\ X(0) = X_0 \end{cases} \tag{2.16}$$

As said above, numerical schemes for Eq. (2.16) has been incorporated for the two well-known methods (viz. Euler Maruyama and Milstein) as below:

Euler-Maruyama Method

We take a time discrete approximation of the Stochastic Differential Equation (SDE) (Higham 2001)

$$\begin{cases} dX(t) = a(t, X)dt + b(t, X)dW_t \\ X(c) = X_c \end{cases} \tag{2.17}$$

Then the approximation scheme for Euler-Maruyama may be represented as follows (Sauer 2012)

$$\begin{aligned} &X_0 = w_0 \\ &w_{i+1} = w_i + a(t_i, w_i)\Delta t_{i+1} + b(t_i, w_i)\Delta W_i \end{aligned} \tag{2.18}$$

where, X_c is the value of X at $t = c$,

$$\begin{aligned} \Delta t_{i+1} &= t_{i+1} - t_i, \\ \Delta W_{i+1} &= W(t_{i+1}) - W(t_i). \end{aligned}$$

We define $N(0,1)$ be the normal distribution and each random number ΔW_i is computed as $\Delta W_i = z_i \sqrt{\Delta t_i}$, where, $z_i \in N(0,1)$.

The obtained set $\{w_0, w_1, ..., w_n\}$ is an approximation realization of the stochastic solution $X(t)$ which depends on the random number z_i that were chosen. Since, W_t is a stochastic process, each realization will be different and so will our approximations.

Milstein Method

The approximation scheme of Milstein method for Eq. (2.17) may be written in the following way (Sauer 2012)

$$X_0 = w_0$$
$$w_{i+1} = w_i + a(t_i, w_i)\Delta t_{i+1} + b(t_i, w_i)\Delta W_i + \frac{1}{2} b(t_i, w_i)\frac{\partial b}{\partial x}(t_i, w_i)(\Delta W_i^{\,2} - \Delta t_i) \tag{2.19}$$

2.4 Theory of Wavelet

A wave is generally defined as disturbance in time or space. For example sinusoid waves are periodic, which repeat after a finite interval of time. But a wavelet may be defined as a wave-like oscillation with amplitude that begins at zero, increases, and then decreases back to zero, that is the value of amplitude over the whole domain is zero. It is a tool for analysis of transient, non-stationary, or time-varying phenomena. It has the oscillating wavelike characteristic but also has the ability to allow simultaneous time and frequency analysis.

Discrete orthogonal wavelets are family of functions with compact support which form a basis on a bounded domain. The orthogonal wavelet family may be defined by a set of L filter coefficients $\{a_l : l = 0, 1, ..., L-1\}$, where L is an even integer. Two fundamental scale equations in wavelet theory are defined as

$$\phi(x) = \sum_{l=0}^{L-1} a_l \phi(2x - l) \tag{2.20}$$

$$\psi(x) = \sum_{l=0}^{L-1} (-1)^l a_{1-l} \phi(2x - l) \tag{2.21}$$

where $\phi(x)$ and $\psi(x)$ are the scaling and wavelet functions with fundamental support over finite intervals $[0, L-1]$ and $[1 - L/2, L/2]$, respectively. These equations may be used to determine the value of the scaling and wavelet functions at dyadic points $x = n/2^J$, $n = 0,1,...$

The scaling functions at resolution level J may be defined as follows (Avudainayagam & Vani 2000)

$$\phi_{J,k}(x) = 2^{J/2}\phi(2^J x - k), \quad k \in Z \tag{2.22}$$

2.4.1 Multi Resolution Analysis

Multiresolution analysis is a sequence of successive approximation spaces $V_j,\ j \in Z$ which satisfies the following properties (Daubechies 1993)

(a) $\cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$ (2.23)

(b) $\bigcup_{n \in Z} V_n$ is dense in $L^2(R)$ (2.24)

(c) $\bigcup_{n \in Z} V_n = \{0\}$ (2.25)

(d) Let us consider an arbitrary function $f(x) \in V_n$ then $f(2x) \in V_{n+1}$ (2.26)

(e) For an arbitrary function $f(x) \in V_0$ we have $f(x-k) \in V_{n+1},\ k \in Z$ (2.27)

Let us consider V_j and W_j be the subspaces of square-integrable functions on real line, $L^2(R)$. Then the linear spans are $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$ and $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, $k \in Z$. Now we may write

$$V_{j+1} = V_j \oplus W_j \tag{2.28}$$

The above relations may be generalized as follows

$$V_0 \subset V_1 \subset \cdots \subset V_j \subset V_{j+1} \tag{2.29}$$

and

$$V_{j+1} = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_j \tag{2.30}$$

where, $\oplus$ denotes the orthogonal direct sum.

2.4.2 Daubechies Wavelet

The wavelet filter coefficients a_l can be derived by (Daubechies 1993) to produce scaling and wavelet functions with the following properties

$$\int_{-\infty}^{\infty} \phi(x)dx = 1 \tag{2.31}$$

$$\sum_{l=0}^{L-1} a_l = 2 \tag{2.32}$$

$$\int_{-\infty}^{\infty} \phi(x)\phi(x-k)dx = \delta_{0,k}, \quad k \in Z \tag{2.33}$$

$$\int_{-\infty}^{\infty} x^k \psi(x)dx = 0, \quad k = 0, 1, ..., L/2-1 \tag{2.34}$$

It may be noted that the property (2.34) is equivalent to the elements of the set $\{1, x, ..., x^{L/2-1}\}$ are linear combinations of $\phi(x-k)$, integer translates of $\phi(x)$. One may have the exact expression as (Chen et al., 1996):

$$\sum_{l=-\infty}^{\infty} l^n \phi(x-l) = x^n + \sum_{j=1}^{n} (-1)^j \binom{n}{j} M_j^{\phi} x^{n-j}, \quad n = 0, 1, ..., L/2-1 \tag{2.35}$$

In Eq. (2.35), M_j^{ϕ} denotes the *j*th moment of $\phi(x)$, which can be computed by the following recursive relation

$$\begin{aligned} M_j^{\phi} &= \int_{-\infty}^{\infty} x^k \phi(x) dx \\ &= \frac{1}{2^{k+1}-2} \sum_{i=0}^{L-1} \sum_{l=1}^{k} \binom{k}{l} a_i i^l M_{k-l}^{\phi} \end{aligned} \tag{2.36}$$

with initial condition $M_0^{\phi} = 1$.

For Daubechies wavelets, we have the following orthogonal properties

$$\int_{-\infty}^{\infty} \phi_{j,k}(x) \phi_{j,l}(x) dx = \delta_{k,l} \tag{2.37}$$

$$\int_{-\infty}^{\infty} \psi_{j,k}(x) \psi_{j,m}(x) dx = 0 \tag{2.38}$$

$$\int_{-\infty}^{\infty} \phi_{j,k}(x) \psi_{j,m}(x) dx = 0 \tag{2.39}$$

where $k = 1, 2, ..., L-2.$

Using the above relations, we may obtain the following derivatives and integrals (Chen et al., 1996)

$$\phi^{(n)}(x) = \frac{d^n \phi(x)}{dx^n} \tag{2.40}$$

$$\theta_n(x) = \underbrace{\int_0^x \int_0^{y_n} \cdots \int_0^{y_2}}_{n-\text{tuple}} \phi(y_1) dy_1 \cdots dy_{n-1} dy_n \tag{2.41}$$

$$M_k^m(x) = \int_0^x y^m \phi(y-k) dy \tag{2.42}$$

$$\Gamma_k^n(x) = \int_0^x \phi^{(n)}(y-k) \phi(y) dy \tag{2.43}$$

$$\Lambda_k^{m,n}(x) = \int_0^x y^m \phi^{(n)}(y-k) \phi(y) dy \tag{2.44}$$

$$\Upsilon_k^{m,n}(x) = \int_0^x y^m \theta_n(y-k) \phi(y) dy \tag{2.45}$$

$$\Omega_{j,k}^{m,n}(x) = \int_0^x \phi(y)\phi^{(m)}(y-j)\phi^{(n)}(y-k)dy \tag{2.46}$$

Here, $j,k \in Z$, $m,n,x \in Z - Z^{-}$, and Z, Z^{-} denote the sets of integer and negative integers respectively.

Chapter 3

Numerical Methods

Chapter 3
Numerical Methods

3.1 Interval and Fuzzy Arithmetic (Limit method)

In view of the literature review of the previous chapter, we may see that the parameters involved in practical systems may be vague or uncertain. These parameters are considered as interval. Then the intervals are converted into crisp form by symbolic parameterization and interval arithmetic has been proposed. The proposed interval arithmetic is now extended for fuzzy numbers and fuzzy arithmetic has been proposed and discussed below.

3.1.1 Proposed Interval Arithmetic

The traditional interval arithmetic sometimes difficult to use. When more number of computations are involved then the process become difficult to handle and the uncertainty rises. It may also be difficult to formulate the methods in general. Here the traditional interval arithmetic have been redefined and proposed.

Let us consider two intervals $[\underline{x}, \overline{x}]$ and $[\underline{y}, \overline{y}]$ then the traditional interval arithematic may be represented in an alternate form as below.

If all the values of the interval are in R^+ or R^- then the arithmetic rules may be written as

(1)
$$[\underline{x}, \overline{x}] + [\underline{y}, \overline{y}] = [\min\{\lim_{n\to\infty} l_1 + \lim_{n\to\infty} l_2, \lim_{n\to 1} l_1 + \lim_{n\to 1} l_2\}, \max\{\lim_{n\to\infty} l_1 + \lim_{n\to\infty} l_2, \lim_{n\to 1} l_1 + \lim_{n\to 1} l_2\}] \quad (3.1)$$

(2)
$$[\underline{x}, \overline{x}] - [\underline{y}, \overline{y}] = [\min\{\lim_{n\to\infty} l_1 - \lim_{n\to 1} l_2, \lim_{n\to 1} l_1 - \lim_{n\to\infty} l_2\}, \max\{\lim_{n\to\infty} l_1 - \lim_{n\to 1} l_2, \lim_{n\to 1} l_1 - \lim_{n\to\infty} l_2\}] \quad (3.2)$$

(3)
$$[\underline{x}, \overline{x}] \times [\underline{y}, \overline{y}] = [\min\{\lim_{n\to\infty} l_1 \times \lim_{n\to\infty} l_2, \lim_{n\to 1} l_1 \times \lim_{n\to 1} l_2\}, \max\{\lim_{n\to\infty} l_1 \times \lim_{n\to\infty} l_2, \lim_{n\to 1} l_1 \times \lim_{n\to 1} l_2\}] \quad (3.3)$$

(4)
$$[\underline{x}, \overline{x}] \div [\underline{y}, \overline{y}] = [\min\{\lim_{n\to\infty} l_1 \div \lim_{n\to 1} l_2, \lim_{n\to 1} l_1 \div \lim_{n\to\infty} l_2\}, \max\{\lim_{n\to\infty} l_1 \div \lim_{n\to 1} l_2, \lim_{n\to 1} l_1 \div \lim_{n\to\infty} l_2\}] \quad (3.4)$$

where $l_1 = \underline{x} + \dfrac{\overline{x} - \underline{x}}{n}$,

$\lim_{n\to\infty} l_1 = \lim_{n\to\infty} \underline{x} + \dfrac{\overline{x}-\underline{x}}{n} = \underline{x}$, $\lim_{n\to 1} l_1 = \lim_{n\to 1} \underline{x} + \dfrac{\overline{x}-\underline{x}}{n} = \overline{x}$;

$l_2 = \underline{y} + \dfrac{\overline{y}-\underline{y}}{n}$,

$\lim_{n\to\infty} l_2 = \lim_{n\to\infty} \underline{y} + \dfrac{\overline{y}-\underline{y}}{n} = \underline{y}$, $\lim_{n\to 1} l_2 = \lim_{n\to 1} \underline{y} + \dfrac{\overline{y}-\underline{y}}{n} = \overline{y}$.

One may obseve that if we consider an interval which includes 0, then we will have at least one solution which is undefined in division operation. For example, if we consider two intervals [-1, 2] and [-3, 1], and divide [-1, 2] by [-3, 1], then we have a solution where 0 divides 0, which is not defined in general. Hence, 0 has not been considered in the above proposed arithmetic. However this difficulty is a great challenge to the interval/fuzzy researchers. In the book (Moore et al. 2014) also the same limitation has been discussed, where $0 \notin Y$ (Y is the divisor).

3.1.2 Proposed Fuzzy Arithmetic

Considering the above proposed interval arithmetic, we may extend it for the fuzzy cases too. One may note that the fuzzy number may easily be transformed into interval form and then the above interval arithmetic may be applied. Following paragraph describes the fuzzy arithmetic.

As such, a triangular fuzzy number $\tilde{A} = [a^L, a^N, a^R]$ may be represented into ordered pair (interval) form by using $\alpha-$cut as below

$$\begin{aligned}\tilde{A} = [a^L, a^N, a^R] &= [a^L + (a^N - a^L)\alpha,\ a^R - (a^R - a^N)\alpha] \\ &= \left[\underline{f}(\alpha), \overline{f}(\alpha)\right], \quad \alpha \in [0,1]\end{aligned}$$

Here $\underline{f}(\alpha)$ and $\overline{f}(\alpha)$ are monotonic increasing and decreasing function respectively. Using these functions we have modified the interval arithmetic (α- cut of fuzzy numbers) as below.

(1) $$\begin{aligned}[\underline{x}(\alpha), \overline{x}(\alpha)] + [\underline{y}(\alpha), \overline{y}(\alpha)] = [&\min\{\lim_{n\to\infty} m_1 + \lim_{n\to\infty} m_2, \lim_{n\to 1} m_1 + \lim_{n\to 1} m_2\}, \\ &\max\{\lim_{n\to\infty} m_1 + \lim_{n\to\infty} m_2, \lim_{n\to 1} m_1 + \lim_{n\to 1} m_2\}]\end{aligned} \tag{3.5}$$

(2) $$\begin{aligned}[\underline{x}(\alpha), \overline{x}(\alpha)] - [\underline{y}(\alpha), \overline{y}(\alpha)] = [&\min\{\lim_{n\to\infty} m_1 - \lim_{n\to 1} m_2, \lim_{n\to 1} m_1 - \lim_{n\to\infty} m_2\}, \\ &\max\{\lim_{n\to\infty} m_1 - \lim_{n\to 1} m_2, \lim_{n\to 1} m_1 - \lim_{n\to\infty} m_2\}]\end{aligned} \tag{3.6}$$

(3) $$[\underline{x}(\alpha), \overline{x}(\alpha)] \times [\underline{y}(\alpha), \overline{y}(\alpha)] = [\min\{\lim_{n\to\infty} m_1 \times \lim_{n\to\infty} m_2, \lim_{n\to 1} m_1 \times \lim_{n\to 1} m_2\}, \tag{3.7}$$

$$\max\{\lim_{n\to\infty} m_1 \times \lim_{n\to\infty} m_2, \lim_{n\to 1} m_1 \times \lim_{n\to 1} m_2\}]$$

(4) $$[\underline{x}(\alpha), \overline{x}(\alpha)] \div [\underline{y}(\alpha), \overline{y}(\alpha)] = [\min\{\lim_{n\to\infty} m_1 \div \lim_{n\to 1} m_2, \lim_{n\to 1} m_1 \div \lim_{n\to\infty} m_2\},$$
$$\max\{\lim_{n\to\infty} m_1 \div \lim_{n\to 1} m_2, \lim_{n\to 1} m_1 \div \lim_{n\to\infty} m_2\}] \tag{3.8}$$

where, $m_1 = \underline{x}(\alpha) + \frac{\overline{x}(\alpha) - \underline{x}(\alpha)}{n}$ and $m_2 = \underline{y}(\alpha) + \frac{\overline{y}(\alpha) - \underline{y}(\alpha)}{n}$, $n \in [1, \infty)$.

3.2 System of Interval/Fuzzy Linear Equations

The uncertain parameters which are in intervals may be changed into symbolic crisp form by applying the above proposed transformation and the new representation of uncertain parameters involves mathematical limits. Using the concept of limits we can compute the interval arithmetic. This arithmetic may easily be computed for solving system of linear interval equations.

Let us consider the interval system of linear equations as

$$\begin{aligned} a^I_{11} x_1 + a^I_{12} x_2 + \ldots + a^I_{1n} x_n &= b^I_1 \\ a^I_{21} x_1 + a^I_{22} x_2 + \ldots + a^I_{2n} x_n &= b^I_2 \\ &\vdots \\ a^I_{m1} x_1 + a^I_{m2} x_2 + \ldots + a^I_{mn} x_n &= b^I_m \end{aligned} \tag{3.9}$$

where the coefficient matrix $A = [a^I_{ij}]$, $1 \le i \le m$; $1 \le j \le n$ is a $m \times n$ interval matrix and y^I_i are intervals.

Whereas, in Eq. (3.9) if the coefficient matrix $\tilde{A} = [\tilde{a}_{ij}]$, $1 \le i \le m$; $1 \le j \le n$ and $\tilde{y}_i$ are fuzzy, then the system is called fuzzy system of linear equations.

The compact form of Eq. (3.9) may be presented in the following way,

$$\sum_{j=1}^{n} [a^I_{ij}]\{x_j\} = \{b^I_j\}, i = 1,2,\ldots,m \tag{3.10}$$

By applying proposed method, Eq. (3.10) may be converted to the following crisp form

$$\sum_{j=1}^{n} \left(\underline{a}_{ij} + \alpha_{ij}\right) x_j = \underline{b}_j + \beta_j, \quad i = 1,2,3,\ldots,m \tag{3.11}$$

where

$$\alpha_{ij} = \frac{\overline{a}_{ij} - \underline{a}_{ij}}{k} \text{ and } \beta_j = \frac{\overline{b}_j - \underline{b}_j}{k}, k \in [1, \infty).$$

Now, the crisp system of linear equations (3.11) may be solved symbolically. In this process, we have crisp system of equations where all the elements of coefficient matrix and righ hand side vector are functions of k. Using standard procedures for crisp systems we may find the solution in terms of k. Then we find the limits (viz. $\lim\limits_{k\to\infty}$ and $\lim\limits_{k\to 1}$) of the symbolic solutions. Finally, maximum and minimum of the obtained solutions are taken. As such, the solution vector can be represented as below:

$$([\min\{\lim_{n\to 1} x_1, \lim_{n\to\infty} x_1\}, \max\{\lim_{n\to 1} x_1, \lim_{n\to\infty} x_1\}], [\min\{\lim_{n\to 1} x_2, \lim_{n\to\infty} x_2\}, \max\{\lim_{n\to 1} x_2, \lim_{n\to\infty} x_2\}],$$
$$\ldots, [\min\{\lim_{n\to 1} x_n, \lim_{n\to\infty} x_n\}, \max\{\lim_{n\to 1} x_n, \lim_{n\to\infty} x_n\}])^T.$$

Here the interval system of linear equations may be applied for fuzzy numbers which are converted into interval form using α-cut approach. As mentioned for interval sytem of linear equations, the fuzzy system of linear equations may be solved in the same manner and the solution vector for the corresponding system can be written as

$$([\min\{\lim_{n\to 1} x_1(\alpha), \lim_{n\to\infty} x_1(\alpha)\}, \max\{\lim_{n\to 1} x_1(\alpha), \lim_{n\to\infty} x_1(\alpha)\}], [\min\{\lim_{n\to 1} x_2(\alpha), \lim_{n\to\infty} x_2(\alpha)\},$$
$$\max\{\lim_{n\to 1} x_2(\alpha), \lim_{n\to\infty} x_2(\alpha)\}], \ldots, [\min\{\lim_{n\to 1} x_n(\alpha), \lim_{n\to\infty} x_n(\alpha)\}, \max\{\lim_{n\to 1} x_n(\alpha), \lim_{n\to\infty} x_n(\alpha)\}])^T.$$

Vertex Method

(Dong & Shah 1987) introduced vertex method for interval system of linear equations. Here combinations of the extremes of interval parameters are used. For *N* interval parameters this corresponds to 2^N simulations. Out of all the computations the lowest and highest values are selected. This method is simple to implement but it may become tedious if the number of vertices increases. The number of computation increases exponentially with the increase in number of interval parameters. Further, the vertex method is accurate only when the conditions of continuity are satisfied. Hence, it may not find optima corresponding to parameter combinations which are not on the vertex of the parameter.

It is found that number of computation in our proposed method becomes less as compare to the vertex method. The present method may be applied to solve the uncertain system of linear equations which generally occurs when heat conduction problems are solved using interval/fuzzy finite element method.

To demonstrate the above procedures let us take the following two examples.

Example 1

Let us now take an interval system of equations

$$\begin{bmatrix} [1,3] & [2,5] \\ [4,7] & [3,8] \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} [2,4] \\ [5,6] \end{Bmatrix}$$

Using the proposed transformation for intervals we get,

$$\Rightarrow \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix},$$

where $a_1 = 1+\frac{2}{n}, a_2 = 2+\frac{3}{n}, a_3 = 4+\frac{3}{n}, a_4 = 3+\frac{5}{n}; b_1 = 2+\frac{2}{n}, b_2 = 5+\frac{3}{n};$ which are all now in crisp form but function of n.

Solving the above system in symbolic form we have

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{b_1 a_4 - a_2 b_2}{a_1 a_4 - a_2 a_3} \\ \dfrac{a_1 b_2 - b_1 a_3}{a_1 a_4 - a_2 a_3} \end{Bmatrix}.$$

Using $\lim\limits_{n\to\infty}$ and $\lim\limits_{n\to 1}$ we write

$$\lim_{n\to\infty} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \lim_{n\to\infty} \begin{Bmatrix} \dfrac{b_1 a_4 - a_2 b_2}{a_1 a_4 - a_2 a_3} \\ \dfrac{a_1 b_2 - b_1 a_3}{a_1 a_4 - a_2 a_3} \end{Bmatrix} = \begin{Bmatrix} 4/5 \\ 3/5 \end{Bmatrix} \text{ and}$$

$$\lim_{n\to 1} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \lim_{n\to 1} \begin{Bmatrix} \dfrac{b_1 a_4 - a_2 b_2}{a_1 a_4 - a_2 a_3} \\ \dfrac{a_1 b_2 - b_1 a_3}{a_1 a_4 - a_2 a_3} \end{Bmatrix} = \begin{Bmatrix} -2/11 \\ 10/11 \end{Bmatrix}$$

Where, $\lim\limits_{n\to\infty} \frac{b_1 a_4 - a_2 b_2}{a_1 a_4 - a_2 a_3} = \frac{4}{5}, \lim\limits_{n\to 1} \frac{b_1 a_4 - a_2 b_2}{a_1 a_4 - a_2 a_3} = -\frac{2}{11}; \lim\limits_{n\to\infty} \frac{a_1 b_2 - b_1 a_3}{a_1 a_4 - a_2 a_3} = \frac{3}{5}, \lim\limits_{n\to 1} \frac{a_1 b_2 - b_1 a_3}{a_1 a_4 - a_2 a_3} = \frac{10}{11}.$

Finally, the solution will be

$$\begin{Bmatrix} \min\{\lim\limits_{n\to\infty} x_1, \lim\limits_{n\to 1} x_1\}, \max\{\lim\limits_{n\to\infty} x_1, \lim\limits_{n\to 1} x_1\} \\ \min\{\lim\limits_{n\to\infty} x_2, \lim\limits_{n\to 1} x_2\}, \max\{\lim\limits_{n\to\infty} x_2, \lim\limits_{n\to 1} x_2\} \end{Bmatrix} = \begin{Bmatrix} [-2/11, 4/5] \\ [3/5, 10/11] \end{Bmatrix}.$$

If we use vertex method for this problem then we have to solve 2^6 combinations of crisp matrix equation and so we have to solve those many systems of equations etc. But here, when we convert the original system into symbolic form then we have to solve only one system although we have to take limits as defined above and finally the maximum and minimum. It is true that for one addition or subtraction it seems that traditional arithmetic will have less labour compared to the proposed procedure. But for actual problems the computations become less.

Example 2

$$x=[0,10],\ y=[0,10],\ x-y-z=0.$$

If we now solve this system by traditional method, we get

$$x-y-z=0$$
$$\Rightarrow [0,10]-[0,10]-z=0$$
$$\Rightarrow z=[0,10]-[0,10]=[-10,10]$$

Now, using present method we may have

$$x=0+\frac{10-0}{k}=\frac{10}{k},\ y=0+\frac{10-0}{k}=\frac{10}{k};\ k\in[1,\infty)$$

$$x-y-z=0$$
$$\Rightarrow z=x-y$$
$$=\frac{10}{k}-\frac{10}{k}=0$$

Taking $\lim_{k\to\infty}$ and $\lim_{k\to1}$ for z, we get $\lim_{k\to\infty} z=0$ and $\lim_{k\to1} z=0$. Hence the solution in this case is [0, 0].

One may see that, by taking one-sided limits (for $n=1$ and $n=\infty$) they may exist (it may be infinite, but still exist), while the corresponding crisp system has singular coefficient matrix. So, by using the limit construction, the algorithm can handle this more general case. This is an advantage over other algorithms. Further it has been noted that the solution such as $x=10$, $y=0$ and $z=10$ excludes from the interval [0, 0] and we may say that the above method is not always conservative.

In the next section we will discuss about the fuzzy finite element method.

3.3 Fuzzy Finite Element Method (FFEM)

If we have parameters as well as initial and boundary conditions as uncertain, the governing differential equations become uncertain. Accordingly, the uncertainties are considered as intervals/fuzzy and interval/fuzzy finite element method have been developed. Considering interval/fuzzy field variables and involve parameters we get interval/fuzzy element properties in terms of interval/fuzzy matrices. Then those matrices are assembled and global matrix is obtained. Further boundary conditions are imposed which may be interval/fuzzy. Taking both the global matrices and boundary conditions we get a system of algebraic equations which are either system of simultaneous equations or eigenvalue problem. As there involve uncertainties (i.e. interval/fuzzy), interval/fuzzy system of simultaneous equations or eigenvalue problems are obtained. The system of interval/fuzzy algebraic equations are investigated by using the above interval/fuzzy arithmetic.

In Figure 3.1, a schematic diagram has been presented for modified fuzzy finite element procedure, which gives the basic idea to encrypt the process of modified fuzzy finite element method. As we have modified the usual interval/fuzzy arithmetic in the finite element method, we have named it as modified fuzzy finite element method. It involves three steps such as input, output and hidden layer. In the input step we have considered uncertain parameters and field variables. These uncertain parameters are taken as fuzzy. In hidden layer, element properties are obtained by using various fuzzy parameters. The element properties and fuzzy stiffness matrices are assembled and finally global stiffness matrices are developed. Further initial and boundary conditions are imposed and the transformed fuzzy system of equations is solved through the proposed technique and various sub steps are executed inside the hidden layer. Finally, we get uncertain fuzzy solutions as output, which may be different in type and nature corresponding to the input fuzzy parameters. Here in Figure 3.1 we have considered triangular fuzzy numbers as input parameters. Alpha (α) level representation of two fuzzy sets $\tilde{X}$ and $\tilde{Y}$ with their triangular membership functions for fuzzy arithmetic operation (Nayak & Chakraverty 2013) is shown in Figure 3.1. Deterministic value is obtained for α_4-level of fuzzy sets whereas for α_1, α_2 and α_3 level we get different interval values. If we consider the α value as zero then deterministic interval lies on X- axis. The output may be generated by considering all possible combinations of the alpha levels.

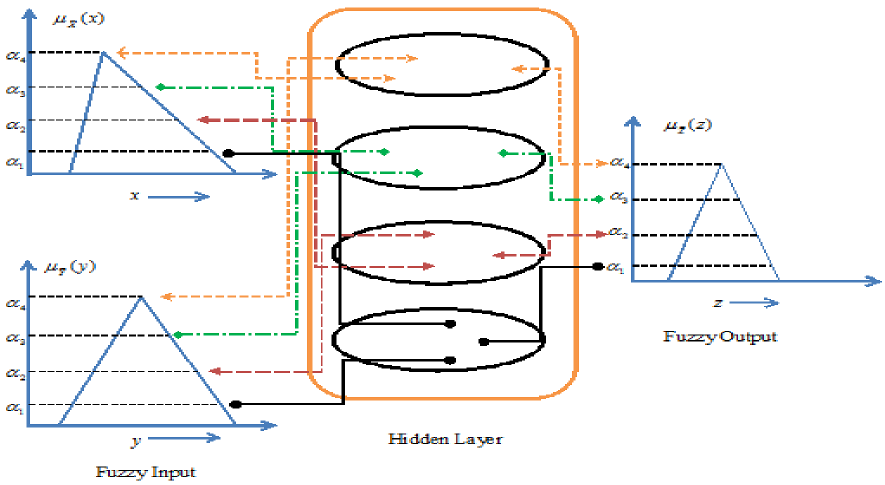

Figure 3.1. Model diagram of the fuzzy finite element procedure

3.4 Fuzzy Euler Maruyama and Milstein Methods

Let us consider stochastic differential equation (2.17) with fuzzy parameters as

$$\begin{cases} d\tilde{X}(t) = \tilde{a}(t, X)dt + \tilde{b}(t, X)dW_t \\ \tilde{X}(0) = \tilde{X}_0 \end{cases} \tag{3.12}$$

As said above, numerical schemes for Eq. (3.12) has been incorporated for the two well-known methods (viz. Euler Maruyama and Milstein) as below:

<u>Fuzzy Euler Maruyama Method</u> (FEMM)

We take a time discrete approximation of an Ito process for the Stochastic Differential Equation (SDE) (Higham 2001) with fuzzy parameters

$$\begin{cases} d\tilde{X}(t) = \tilde{a}(t, X)dt + \tilde{b}(t, X)dW_t \\ \tilde{X}(c) = \tilde{X}_c \end{cases} \tag{3.13}$$

Then the approximation scheme for fuzzy Euler-Maruyama may be represented as follows (Sauer 2012)

$$\begin{aligned} &\tilde{X}_0 = \tilde{w}_0 \\ &\tilde{w}_{i+1} = \tilde{w}_i + \tilde{a}(t_i, w_i)\Delta t_{i+1} + \tilde{b}(t_i, w_i)\Delta W_i \end{aligned} \tag{3.14}$$

where, $\tilde{X}_c$ is the value of $\tilde{X}$ at $t = c$,

$$\Delta t_{i+1} = t_{i+1} - t_i,$$
$$\Delta W_{i+1} = W(t_{i+1}) - W(t_i).$$

Define $N(0,1)$ be the normal distribution and each random number ΔW_i is computed as

$\Delta W_i = z_i\sqrt{\Delta t_i}$

where, $z_i \in N(0,1)$.

Here the obtained set $\{\tilde{w}_0, \tilde{w}_1, ..., \tilde{w}_n\}$ is an approximation realization of the stochastic solution $X(t)$ which depends on the random numbers z_i that were chosen. Since, W_t is a stochastic process, each realization will be different and so will our approximations.

<u>Fuzzy Milstein Method</u> (FMM)

Similarly, the approximation scheme of fuzzy Milstein method for Eq. (3.13) may be written in the following way

$$\begin{aligned} &\tilde{X}_0 = \tilde{w}_0 \\ &\tilde{w}_{i+1} = \tilde{w}_i + \tilde{a}(t_i, w_i)\Delta t_{i+1} + \tilde{b}(t_i, w_i)\Delta W_i + \frac{1}{2}\tilde{b}(t_i, w_i)\frac{\partial \tilde{b}}{\partial x}(t_i, w_i)(\Delta W_i^{\ 2} - \Delta t_i) \end{aligned} \tag{3.15}$$

Above methods have been discussed in the following subsections.

Chapter 4

Heat Conduction Problems

The contents of this chapter have been published in the following Journal/Book/Conference:

1. S. Chakraverty and S. Nayak, Fuzzy finite element method for solving uncertain heat conduction problems, ***Coupled Systems Mechanics***, 2012, Vol. 1, No. 4, 345-360;
2. S. Chakraverty and S. Nayak, Fuzzy finite element method in diffusion problems, Mathematics of Uncertainty Modeling in the Analysis of Engineering and Science Problems, ***IGI Publication***, USA, edited by S. Chakraverty, 2013;
3. S. Nayak and S. Chakraverty, Fuzzy finite element approach for solving uncertain heat conduction problem, National Conference on Mathematics of Soft Computing (NCMSC-2012), National Institute of Technology Calicut, 5-7 July, 2012.

Chapter 4

Heat Conduction Problem

Heat transfers from a body having higher temperature to lower temperature. Conduction is a type of heat transfer where molecular agitation takes place within a material without any motion of the material as a whole. The modelling of heat conduction for different materials involves various coefficients and constants viz. thermal conductivity and specific heat capacity etc. Generally these parameters are uncertain and hence the values of these parameters are considered as fuzzy. As such, heat conduction within a cylindrical rod has been investigated here. Fuzzy finite element method (of chapter 3) has used to solve coupled heat conduction problem. Corresponding differential equation for the said problem is converted into fuzzy algebraic equations by using fuzzy finite element method. As a result, we get a coupled set of fuzzy linear equations. To handle the coupled set of fuzzy linear equations, two techniques viz. fuzzy iteration and fuzzy eigenvalue methods are discussed. Obtained results have been compared in special cases.

4.1 Uncertain Heat Conduction

Uncertainty plays a vital role in various fields of science and engineering. These uncertainties occur due to incomplete data, impreciseness, vagueness, experimental error and different operating conditions influenced by the system. Different authors proposed various methods to handle uncertainty. They have used probabilistic or statistical method as a tool to operate uncertain parameters. But in this process we need more number of observed data or experimental results to analyse the problems. Practically it may not be possible something to get a large number of data because it needs more number of experiments to perform. So instead of probabilistic or statistical approach we may use interval or fuzzy parameters to handle uncertainty which require less number of data. In general traditional interval/fuzzy arithmetic are complicated to manage the problem rigorously. Accordingly, we have used interval/fuzzy arithmetic (of Chapter 3) to handle such difficulty which are simple and efficient. The main aim of this investigation is not the particular problem considered rather we shall concentrate on how the proposed methods can be applied to various heat conduction problems. Modelling of heat conduction problems may be represented by different types of differential equations. The governing differential equations although are solved earlier by various authors using exact methods (Carlslaw & Jaeger 1986; Liu et al. 1986; Bondarev 1997; Monte 2000) and numerical methods (Peterson, Richard 1999; Iijima 2004; Liu & Cheng 2006). Also probabilistic and statistical methods have also been introduced to calculate variability of involved parameters through the heat and mass transfer model. To handle such variability, several probabilistic methods have been introduced. As such, Monte Carlo method is used to solve heat and mass propagation problem. It essentially involves a large number of process samples which are obtained by numerically solving the problem for artificially generated random parameter samples. Monte Carlo method has been used to analyse thermal food processes with variable parameters (Wang et al. 1991; Varga et al. 2000; Caro-Corrales et al. 2002; Demir et al. 2003; Halder et al. 2007; Laguerre & Flick 2010). (Deng & Liu 2002) implemented Monte Carlo method to solve the direct bio heat transfer problems. They have demonstrated the bio heat transfer problem with transient or space-dependent boundary conditions, blood perfusion, metabolic rate, and volumetric heat source for tissue.

On the other hand various numerical techniques are proposed viz. Finite Difference Method (FDM), Finite Volume Method (FVM) and Finite Element Method (FEM) (Magnus & Achi 2011; Muhieddine et al. 2009). (Magnus & Achi 2011) used finite difference method to model and solve the governing ground water flow rates, flow direction and hydraulic heads through an aquifer. (Muhieddine et al. 2009) described one dimensional phase change problem. They have used vertex centred finite volume method to solve the problem.

(Edward & Robert 1966) used FEM to solve heat conduction problem and analyse it. A non-iterative, finite element-based inverse method for estimating surface heat flux histories on thermally conducting bodies developed by (Ling et al. 2003). They considered both linear and non-linear problems, and sequentially minimizes the least square error norm between corresponding sets of measured and computed temperatures. Further, (Onate et al. 2006) used Galerkin FEM for convective–diffusive problems with sharp gradients using finite calculus. In view of the above literatures, it reveals that the traditional finite element method may easily be used where the parameters or the values are exact that is in crisp form. But in actual practice the values may be in a region of possibility or we can say the values are uncertain. These uncertain parameters give uncertain model predictions. Although the uncertainty may be reduced by appropriate experiments but still it may also give the variability in the parameters. Then finite element perturbation method is used by (Nicolaï & Baerdemaeker 1993) and (Nicolaï et al., 2000) for heat conduction problem with uncertain physical parameters. Further (Nicolaï, Verboven, et al., 1999; Nicolaï, Verlinden, et al., 1999) found the temperature in heat conduction problem for randomly varying parameters with respect to time. These authors have used a variance propagation technique to calculate the mean and covariance of the temperatures.

It is very difficult to get a large number of experimental data, so we need an alternative method in which we may handle the uncertainty considering few experimental data. In this context (Zadeh 1965) proposed an alternate idea i.e. fuzzy set theory to handle uncertain values. Hence we need the help of interval/fuzzy analysis for handling these types of data. The direct implementation of interval or fuzzy becomes more complex and the computation is also a difficult task. So to avoid such difficulty different authors tried new techniques to handle such difficulty. (Dong & Shah, 1987) proposed vertex method for computing functions of fuzzy variables. (Dong & Wong 1987) used Fuzzy Weighted Average Method (FWAM). (Yang et al. 1993) discussed the calculation of functions with fuzzy numbers. They developed methods which require less computation than the FWAM. (Klir 1997) revised fuzzy arithmetic by considering the relevant requisite constraints. A transformation method based on the concept of α-cut has been presented by (Hanss 2002), where the fuzzy arithmetic is reduced to the interval computation.

Here we have taken the parameters as fuzzy and then it is converted into interval by proposed method (of Chapter 3). The interval values are transformed into crisp form using a proposed transformation. Then we present the traditional finite element procedure (Muhanna & Mullen 2001; Neumaier 1990; Kulpa et al. 1998) for solving the problem by taking these parameters in interval. Next interval/fuzzy finite element technique is described for the said problem. Here we operate interval/fuzzy parameters through finite element method using proposed arithmetic. Fuzzy finite element method results into a set of algebraic equations. These set of algebraic equations will be a fuzzy system of simultaneous equations in this case. (Matinfar et al. 2008) used householder decomposition method to solve fuzzy linear equations and they considered only the right hand side column vector as fuzzy and solved some example problems. For the fuzzy coefficient matrix A, (Panahi et al. 2008) obtained lower triangular and upper triangular matrix separately. (Senthilkumar & Rajendran 2011) considered symmetric coefficient matrix to solve Fuzzy Finite Linear System (FFLS) of equations. Here they decomposed the coefficient matrix by using Cholesky method. However (Vijayalakshmi & Sattanathan 2011) introduced Symmetric times Triangular (ST) decomposition procedure to solve fully fuzzy system of linear equations and a simple arithmetic has also presented to handle fuzzy system of linear equations. Further, (Nicolai et al. 2011) investigated the uncertain solution of heat conduction problem and gave a good comparison between response surface method and other methods. Finite element method in the present problem turns into a system of coupled equations. These equations are difficult to solve directly. So we have used two different techniques to solve and these are iteration and eigenvalue methods respectively. Below these methods are discussed in detail. We have used a new form of fuzzy finite element method to solve the uncertain heat conduction problem. Finally, obtained results have been presented and compared in special cases.

4.2 Finite Element Formulation

The principle of conservation of energy says that sum of input energy and energy generated is equal to the sum of increase in energy and the output energy. This principle may be applied to heat transfer problems. Generally heat conduction may be classified in two ways; these are steady and unsteady state heat conduction.

Consider a time dependent heat transfer problem in a three dimensional anisotropic solid Ω bounded by a surface Γ. Then the governing differential equation for this problem is given by

$$-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)+Q=\rho c\frac{\partial T}{\partial t} \tag{4.1}$$

where

q_x, q_y and q_z are components of heat flow rate per unit area in cartesian coordinates, Q is the internal heat generation per unit volume, ρ is the density and c is the specific heat capacity. The finite element formulation for the above governing differential equation in crisp form may be represented as

$$[C]\left\{\frac{dT}{dt}\right\}+[K]\{T\}=\{F_Q\}+\{F_g\} \tag{4.2}$$

where [.] and {.} represents the matrix and vector respectively.

If N is the shape function then the terms defined above in Eq. (4.2) are as follows

$$
\begin{aligned}
[C] &= \int_{\Omega^{(e)}} \rho c \{N\}[N] d\Omega \\
[K_c] &= \int_{\Omega^{(e)}} [B]^T [k][B] d\Omega \\
[K_h] &= \int_{\Omega^{(e)}} \rho c \{N\}[N] d\Omega \\
\{F_Q\} &= \int_e Q\{N\} d\Omega \\
\{F_g\} &= -\int_\Gamma q.\hat{n}\{N\} d\Gamma
\end{aligned} \tag{4.3}
$$

The general form of the governing differential equation for three dimensional time dependent heat conduction problem, may be written as

$$
K_x \frac{\partial^2 T}{\partial x^2} + K_y \frac{\partial^2 T}{\partial y^2} + K_z \frac{\partial^2 T}{\partial z^2} + Q = c\rho \frac{\partial T}{\partial t} \tag{4.4}
$$

The domain which satisfies the above equation is discretized into some finite numbers of elements. For every element we find the approximated function and then we get stiffness matrices for each element. For finding the stiffness matrices of complete domain we need to assemble each element. There are two different stiffness matrices (for each element) used for the said problem and these are capacitance and conductance matrices.

Considering one dimensional heat conduction problem, the capacitance and conductance matrix for each element may be represented as $[C^{(e)}] = \frac{c\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $[K^{(e)}] = \frac{kA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ respectively, where k is the thermal conductivity, c is specific heat, ρ is density, A is area of cross section and L is the length. The global stiffness matrices for capacitance and conductance for n elements are given as

$$
[C] = \frac{c\rho AL}{6}\begin{bmatrix}
2 & 1 & 0 & \cdots & \cdots & 0 \\
1 & 4 & 1 & 0 & 0 & \vdots \\
0 & 1 & 4 & 1 & 0 & \vdots \\
\vdots & 0 & 1 & \ddots & \ddots & 0 \\
\vdots & 0 & 0 & \ddots & 4 & 1 \\
0 & \cdots & \cdots & 0 & 1 & 2
\end{bmatrix}_{(n+1)\times(n+1)}
$$

and

$$
[K] = \frac{kA}{L}\begin{bmatrix}
1 & -1 & 0 & \cdots & \cdots & 0 \\
-1 & 2 & -1 & 0 & 0 & \vdots \\
0 & -1 & 2 & -1 & 0 & \vdots \\
\vdots & 0 & -1 & \ddots & \ddots & 0 \\
\vdots & 0 & 0 & \ddots & 2 & -1 \\
0 & \cdots & \cdots & 0 & -1 & 1
\end{bmatrix}_{(n+1)\times(n+1)}
$$

respectively.

It may be noted that the above capacitance and conductance matrices are easy to generate for crisp problems. But, when we take the uncertainty in term of interval or fuzzy then these matrices are called interval/fuzzy matrices. As said earlier, researchers proposed few methods to solve these but these are done for some special cases. Here the newly proposed method (Chapter 3) of uncertain value is generated and used in a simple problem of heat conduction. The main aim of this chapter is not the example problem taken but the proposed new method which can be used for other complicated problem of heat conduction.

It is worth mentioning that the parameters involved in Eqs. (4.1), (4.2) and (4.4) are considered as interval. Moreover the solution T is a vector and its components are (T_i, i=n+1) where these components are nothing but the intervals. As such we get the solution T in the form of hyper cuboid of dimension $n + 1$. Taking the help of above discussed method using finite element method we may investigate the hyper cuboid. Considering the proposed interval/fuzzy finite element method, a simple heat conduction problem has been solved which is discussed in the subsequent sections.

4.3 Numerical Modeling and Results

Heat conduction may be divided into two category i.e. steady and unsteady state. In steady state heat conduction problem we do not consider the time dependent term whereas unsteady state heat conduction is time dependent. These problems may be solved numerically by using numerical method such as finite difference and finite element methods etc. Here finite element method has been considered to solve an example problem. Again the governing differential equations for heat conduction problems involve uncertainty. Hence to solve uncertain governing differential equations we have used interval/fuzzy finite element method (Chapter 3). After using interval/fuzzy finite element method we will get a system of interval/fuzzy ordinary differential equations. We have considered two approaches to solve this system of interval/fuzzy ordinary differential equations. These approaches are discussed in the following subsections.

4.3.1 Iterative Method

Consider a general first order time dependent differential equation as follows

$$[\tilde{C}]\{\dot{\tilde{T}}\}+[\tilde{K}]\{\tilde{T}\}=[\tilde{R}(t)] \tag{4.5}$$

where $\tilde{C}, \tilde{K}, \tilde{R}$ are fuzzy.

This can be solved by the method of iteration. The procedure relies on deriving the recursion formula that relates the value $\{\tilde{T}\}$ at one instant of time t to the values of $\{\tilde{T}\}$ at time $t+\Delta t$, where Δt is the time step. The recursion formula make it possible for the solution to be marched out in time, starting from the initial condition at time $t=0$ sec. and continuing step by step until reaching the desired duration.

Let $t_{n+1}=t_n+\Delta t$, where *n=0, 1, 2,…, N.*

Now applying finite difference approximation with a time step *t=1*sec., Eq. (4.5) becomes

$$\left[\tilde{C}\right]\frac{\left\{\tilde{T}\right\}_{n+1}-\left\{\tilde{T}\right\}_{n}}{\Delta t}+\left[\tilde{K}\right]\left\{\tilde{T}\right\}_{n}=\left[\tilde{R}(t)\right]$$

$$or,\ \left\{\tilde{T}\right\}_{n+1}=\Delta t\left[\tilde{R}(t)\right]\left[\tilde{C}\right]^{-1}-\Delta t\left[\tilde{K}\right]\left[\tilde{C}\right]^{-1}\left\{\tilde{T}\right\}_{n}+\left\{\tilde{T}\right\}_{n} \tag{4.6}$$

Using this algorithm we may found the value of $\{\tilde{T}\}$ which converges to the desired values depending upon the relative error of temperatures.

4.3.2 Eigenvalue Method

Eigenvalue method is widely used for second order matrix equations encountered in structural dynamic problems. The same method is extracted for first order differential equations viz. Eq. (4.5) also discussed below.

Consider the first order differential Eq. (4.5)

Substituting $\left\{\tilde{T}\right\}=\left\{\tilde{P}\right\}e^{-\lambda t}$ in Eq. (4.4) and taking $\tilde{R}(t)=0$ we get

$$[[\tilde{K}]-\lambda[\tilde{C}]]\left\{\tilde{T}\right\}=0 \tag{4.7}$$

where $\{\tilde{P}\}$ is a modal vector and λ is a modal decay constant.

Here, Eq. (4.5) is similar to the eigenvalue problem. For $n\times n$ matrices there will be *n* values of λ_i (eigenvalues) and *n* values of $\{\tilde{P}\}_i$ (eigenvectors).

For two different eigenvectors $\{\tilde{P}\}_i$ and $\{\tilde{P}\}_j$ corresponding to two eigenvalues $\tilde{\lambda}_i$ and $\tilde{\lambda}_j$ respectively, we have the orthogonal condition as

$$\begin{aligned}\{\tilde{P}\}_i[\tilde{K}]\{\tilde{P}\}_j\neq 0,\quad i\neq j\\ \{\tilde{P}\}_i[\tilde{C}]\{\tilde{P}\}_j\neq 0,\quad i\neq j\end{aligned} \tag{4.8}$$

For *i=j* we have

$$\{\tilde{P}\}_i[\tilde{K}]\{\tilde{P}\}_i = K_{ii}^*$$
$$\{\tilde{P}\}_i[\tilde{C}]\{\tilde{P}\}_i = C_{ii}^* \tag{4.9}$$

where K_{ii}^* and C_{ii}^* are constants.

Now a transformation $\{\tilde{T}\} = [\tilde{P}]\{\tilde{y}\}$ in Eq. (4.4) is used, where $[\tilde{P}]$ is a fuzzy modal matrix containing the eigenvectors as columns and it becomes

$$[\tilde{C}][\tilde{P}]\{\dot{\tilde{y}}\} + [\tilde{K}][\tilde{P}]\{\tilde{y}\} = \{\tilde{R}\} \tag{4.10}$$

Multiplying $[\tilde{P}]^T$ both the sides of Eq. (4.10), we get

$$[\tilde{P}]^T[\tilde{C}][\tilde{P}]\{\dot{\tilde{y}}\} + [\tilde{P}]^T[\tilde{K}][\tilde{P}]\{\tilde{y}\} = [\tilde{P}]^T\{\tilde{R}\} \tag{4.11}$$

In view of the Eqs. (4.8) and (4.9), Eq. (4.11) will give a system of first order ordinary differential equations which may easily be solved by any standard methods.

4.3.3 Case Study

Let us consider a cylindrical rod whose right end of is provided at a constant temperature $30^\circ C$. At time zero, the entire rod is at a temperature of $30^\circ C$ when a heat source is applied to the left end, bringing the temperature of the left end immediately to $80^\circ C$ and maintaining that temperature indefinitely. Corresponding data for this problem is provided in Table 4.1.

Table 4.1. Fuzzy parameters for conduction in a rod (Test problem)

Parameters	Notations	Normalized value	Fuzzy value (TFN)
Diameter	D	12 mm	12 mm
Length	l	100 mm	100 mm
Density	d	$2700\ kg/m^3$	$2700\ kg/m^3$
Thermal conductivity	k	$200 W/m - ℃$	$[195, 200, 205]$ $W/m - ℃$
Specific heat capacity	c	$900\ J/kg\ - ℃$	$[895, 900, 905]$ $J/kg\ - ℃$

It is very well known from Fick's law of diffusion that the diffusive flux goes from regions of high concentration to regions of low concentration, with a magnitude that is proportional to the concentration gradient. It is said that the diffusive flux tries to attain an equilibrium position. We have taken the said propagation problem where the temperature of the rod propagates and want to stable. By using the iteration scheme we have found that the temperatures become stable after some iterations depending upon the error. The crisp results of iteration method for the said problem are given in the Table 4.2.

Table 4.2. Nodal temperatures of the rod (with crisp variables)

Temperature	Error (0.3%)	Error (0.1%)	Error (0.03%)
T_2	66.0145	66.9731	67.3288
T_3	52.8949	54.2506	54.7536
T_4	41.0097	41.9683	42.3240
Number of iterations	30	42	56

The iterations are taken with a time interval of 1 sec. each. The temperatures converge if we take 30, 42, 56 iterations for relative errors 0.3%, 0.1% and 0.03% respectively. Temperature starts from initial value which is given in the said problem and then transported through the rod and maintain a constant temperature after a certain time. At different relative errors one can see that the temperatures at the nodes (T_2, T_3, T_4) differ with a little margin. But in real practice the parameters used for the said problem are not in crisp form rather in interval/fuzzy. So considering the fuzzy value of involved parameters we may evaluate the nodal temperature. Considering different relative errors the resultant temperature distribution are shown in Table 4.3. When value of c and k both are taken as fuzzy, we observe that the corresponding temperatures changes slightly with the variations of relative errors. Again the uncertainty of temperatures at the stage of stability becomes narrow. If we compare the results of Table 4.3 with Table 4.2, we get the centre value of Table 4.3 very close to the value of the Table 4.2. The time taken for stability is also same if we compare with the time taken in Table 4.2.

In comparison with the vertex method, proposed method may be a better alternative. The number of computations in vertex method is 2^N, where N is the number of fuzzy parameters used and in our approach only two computations are needed. Again the presence of limits become easy to handle. By this we do not have to check all the combinations for the uncertain parameters except for the two combinations.

Table 4.3. Nodal temperatures (with uncertain fuzzy variables)

Temperature		Error (0.3%)	No. of iterations	Error (0.1%)	No. of iterations	Error (0.03%)	No. of iterations
T_2	Left	66.1115	30	67.0332	42	67.3705	56
	Right	66.1259	30	67.0480	42	67.3835	56
	Center	66.1187	30	67.0406	42	67.377	56
T_3	Left	53.0411	30	54.3445	42	54.8188	56
	Right	53.0587	30	54.3628	42	54.8399	56
	Center	53.0499	30	54.35365	42	54.82935	56
T_4	Left	41.1164	30	42.0381	42	42.3710	56
	Right	41.1264	30	42.0485	42	42.3884	56
	Centre	41.1214	30	42.0433	42	42.3797	56

Further, applying the above defined eigenvalue method we may also solve the said problem. The modeled differential equation gives a coupled set of differential equations which may sometimes be difficult to solve. So we have used eigenvalue method which transforms the coupled equations into uncoupled equations. These uncoupled equations give a system of first order linear differential equation which has been solved by using usual method. The fuzzy parameters are incorporated in the differential equation and using proposed method the temperatures are obtained. For different time interval of 20 sec. the nodal temperatures are given in Table 4.4. Fuzzy parameters for the said problem are same as used for Table 4.3. The solutions of the first ordered fuzzy linear differential equations depend on time. Here we have considered the temperatures at time 20, 40, 60, 80 and 100 sec. respectively. Next, we observed that the nodal temperatures get closer with respect to time. As we increase the time, the uncertain width of the solution temperature becomes small which suggest that the stability condition getting better and better by spending more time.

We have given a comparison between the center solutions of fuzzy with the crisp solution in Table 4.5. The center solutions of uncertain fuzzy values and the exact solutions of crisp values at 20, 40, 60, 80 and 100 sec. are presented in Table 4.5. We can easily see that the center and exact values are very close which means the uncertain fuzzy values are giving good results. The fuzzy results are given in Figures 4.1 to 4.3. Here five different membership functions are given for corresponding time 20, 40, 60, 80 and 100 sec. The membership functions getting closer and closer if we move with time. For vertex method we need more complication to solve differential equations. More number of fuzzy parameters will need large number of calculations. To find the bound of uncertainty in vertex method we have to calculate maximum and minimum values out of all the possible solutions.

Table 4.4. Nodal temperatures (for uncertain fuzzy parameters) with different time

Temperature		T_2	T_3	T_4
20 sec.	Left	63.4823	49.3429	38.4935
	Right	63.7662	49.7382	38.7727
	Center	63.62425	49.54055	38.6331
40 sec.	Left	66.7242	53.9212	41.7246
	Right	66.8555	54.1027	41.8541
	Center	66.78985	54.01195	41.78935
60 sec.	Left	67.3286	54.7762	42.329
	Right	67.396	54.8672	42.3946
	Center	67.3623	54.8217	42.3618
80 sec.	Left	67.4414	54.9359	42.4418
	Right	67.4907	55.0011	42.4892
	Center	67.46605	54.9685	42.4655
100 sec.	Left	67.4625	54.9657	42.4629
	Right	67.5072	55.0246	42.5058
	Center	67.48485	54.99515	42.48435

Table 4.5. Comparison of temperatures between exact and centre solution

Temperature		T_2	T_3	T_4
20 sec.	Center	63.62425	49.54055	38.6331
	Exact	63.6291	49.5359	38.6391
40 sec.	Center	66.78985	54.01195	41.78935
	Exact	66.7939	54.0055	41.7947
60 sec.	Center	67.3623	54.8217	42.3618
	Exact	67.3655	54.8139	42.3663
80 sec.	Center	67.46605	54.9685	42.4655
	Exact	67.4689	54.9602	42.4697
100 sec.	Center	67.48485	54.99515	42.48435
	Exact	67.4876	54.9866	42.4884

Whereas, we need only two computations to find the uncertain bound of solutions using the proposed method. Also the differential equation is much easier to handle by the eigenvalue method.

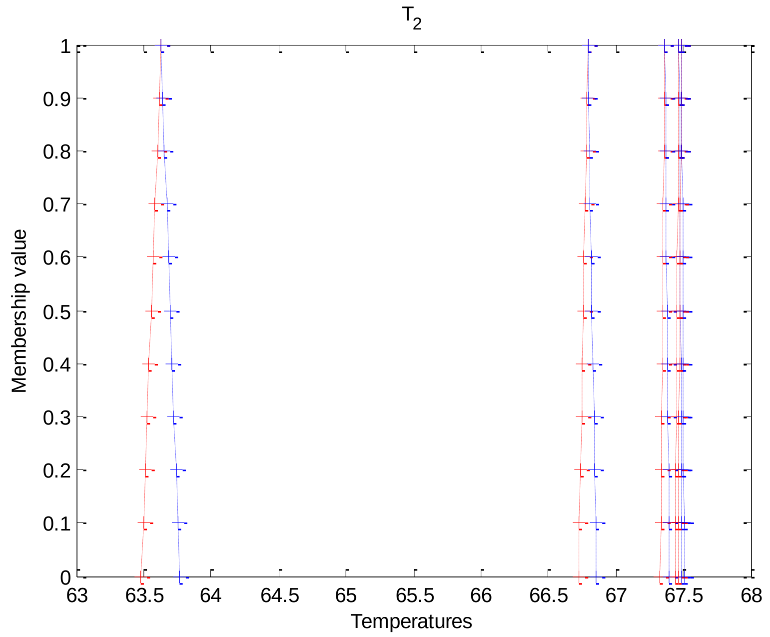

Figure 4.1. Membership function of temperature T_2 at a time difference of 20 sec.

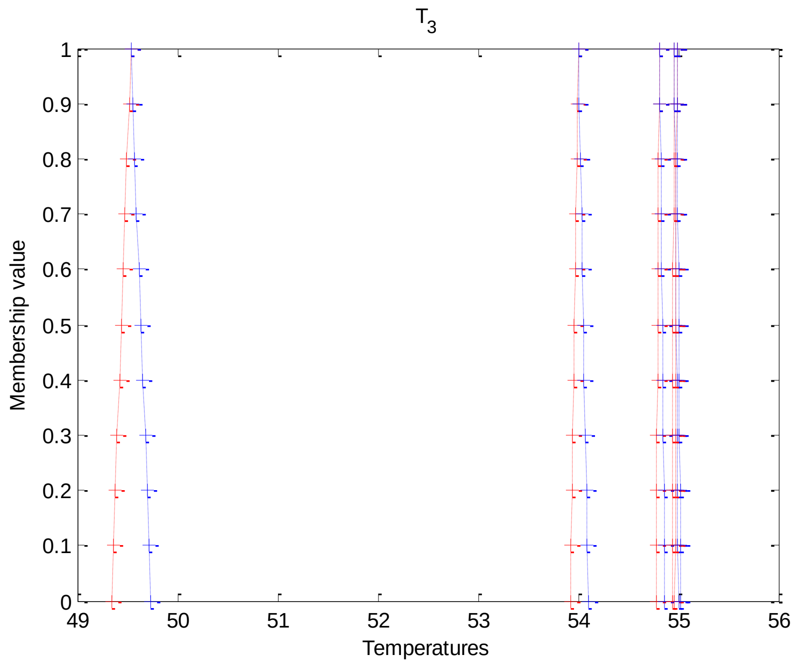

Figure 4.2. Membership function of temperature T_3 at a time difference of 20 sec.

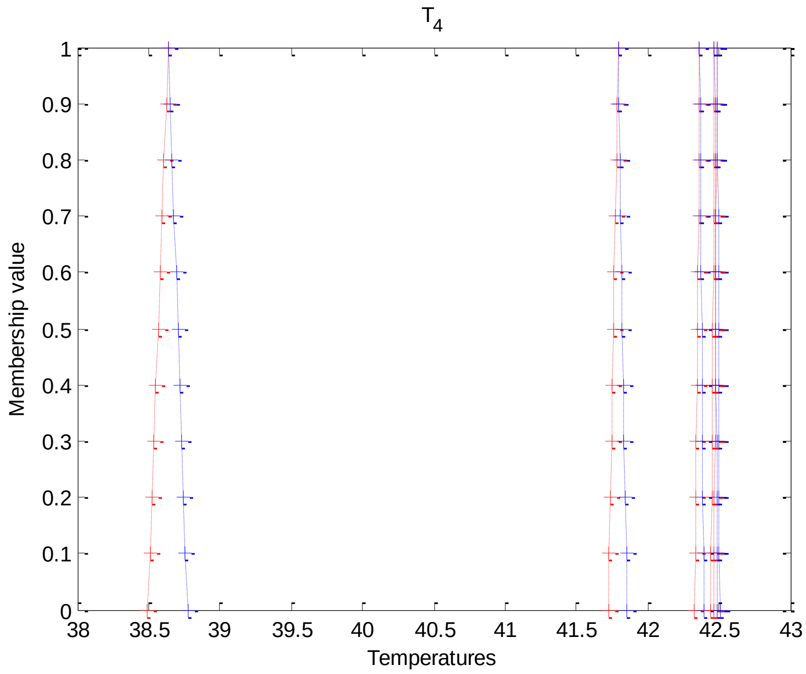

Figure 4.3. Membership function of temperature T_4 at a time difference of 20 sec.

4.4 Conclusion

The main aim of this investigation is to provide an alternative non probabilistic method to handle uncertain parameters involved in the system. The traditional interval arithmetic is transformed into a unique way and a new method is proposed for interval arithmetic. This interval arithmetic is then extended for triangular fuzzy numbers and is used in finite element method. It has been noted that the above interval arithmetic may also be extended for trapezoidal fuzzy numbers using α -cut technique. The resulting fuzzy finite element method is used to solve a test problem. Corresponding differential equation for the said problem is converted into fuzzy algebraic equations by the use of fuzzy finite element method. Hence we get a coupled set of fuzzy linear equations. Due to the difficulty to manage coupled set of fuzzy linear equations we have proposed two techniques viz. fuzzy iteration and fuzzy eigenvalue methods. Obtained results are compared in special cases and the proposed method is found to be better than the well-known vertex method.

Chapter 5

Heat Conduction-Convection Problems

The contents of this chapter have been published in the following journals:

1. S. Nayak and S. Chakraverty, Non-probabilistic approach to investigate uncertain conjugate heat transfer in an imprecisely defined plate, ***International Journal of Heat and Mass Transfer***, 2013, Vol. 67, 445–454;
2. S. Nayak, S. Chakraverty and D. Datta, Uncertain spectrum of temperatures in a non-homogeneous fin under imprecisely defined conduction-convection system, ***Journal of Uncertain Systems***, 2014, Vol. 8, No. 2, 123-135.

Chapter 5

Heat Conduction-Convection Problems

Conjugate heat transfer is a process which involves a coupling of conduction in the solid and convection in the fluid. Various studies have been done in this field by considering only crisp parameters. But we may not ignore the uncertainty involved in this system. So to get more acceptable and reliable results we have considered here the involved parameters as fuzzy. FEMM (Chapter 3) has been used to investigate two different problems viz. non-homogeneous tapered fin and plate. However, FFEM involves the complicated operation of uncertain/fuzzy numbers. So, these fuzzy numbers are first changed into intervals through α -cut and then it is converted into crisp form by a proposed procedure. This crisp representation is used as a tool to handle fuzzy finite element method. Here two types of fuzzy numbers viz. TFN and TRFN have been considered for the investigation. For the above problems MATLAB program has been coded and the distribution of uncertain temeparures are obtained. Different combinations of uncertain parameters are taken and the sensitiveness of these papametrs over the system has also been discussed.

5.1 Non Homogeneous Fin

Heat transfer is a common phenomenon which may be found in various fields of science and engineering. This is actually a multi-dimensional conjugate problem, in which heat conduction takes place not only in the direction orthogonal to the walls (transverse conduction), but also parallel to them (longitudinal conduction). Here conjugate heat transfer refers to a heat transfer process involving an interaction of conduction within a solid body and the convection from the solid surface to fluid moving over the surface. Therefore, a realistic analysis of conjugate heat transfer problems necessitates the coupling of the conduction in the solid and the convection in the fluid. But in actual practice when we deal with conduction-convection system we come across the uncertainty caused due to the involved parameters in the system. These uncertainties occur due to the imprecise data, experimental error and vagueness of involved parameters as mentioned earlier.

In this context (Zadeh 1965) proposed an alternate idea that is fuzzy number to handle uncertain values. This concept has been used here to investigate the problem. The direct implementation of interval or fuzzy becomes more complex and the computation is also a difficult task. To avoid such difficulties various authors have been used different techniques. (Dong & Shah 1987) proposed vertex method for computing functions of fuzzy variables.

(Dong & Wong 1987) used Fuzzy Weighted Average Method (FWAM). (Yang et al. 1993) discussed the calculation of functions with fuzzy numbers. They developed methods which require less computation than the FWAM. Considering the relevant requisite constraints (Klir 1997) revised fuzzy arithmetic. (Hanss 2002) gave a transformation method based on the concept of α-cut where the fuzzy arithmetic is reduced to interval computation.

Uncertain parameters for the present purpose have been addressed in the form of a fuzzy number and using α-cut definition of fuzzy set various intervals are constructed for the specified fuzzy number. The interval values are transformed into crisp form using a proposed transformation. Then we present the traditional finite element procedure (Neumaier 1990; Kulpa et al. 1998; Muhanna & Mullen 2001) for solving the problem by taking these parameters in interval. Next interval/fuzzy finite element technique has been used to investigate the said problem. Here we operate interval/fuzzy parameters through finite element method using proposed arithmetic. Fuzzy finite element method results into a set of algebraic equations. These set of algebraic equations will be a fuzzy system of simultaneous equations in this case. (Matinfar et al. 2008) used householder decomposition method to solve fuzzy linear equations and they considered only the right hand side column vector as fuzzy and solved some example problems. For the fuzzy coefficient matrix A, (Panahi et al. 2008) obtained lower triangular and upper triangular matrix separately. (Senthilkumar & Rajendran 2011) considered symmetric coefficient matrix to solve Fully Fuzzy Linear System (FFLS) of equations. They decomposed the coefficient matrix by using Cholesky method. However, (Vijayalakshmi & Sattanathan 2011) introduced Symmetric times Triangular (ST) decomposition procedure to solve fully fuzzy system of linear equations. (Behera & Chakraverty 2013b) proposed a method to solve fuzzy real system of linear equations by solving two $n \times n$ crisp systems of linear equations. Here the coefficient matrix is considered as real crisp, whereas an unknown variable vector and right hand side vector are considered as fuzzy. The general system is solved by adding and subtracting the left and right bounds of the vectors respectively. Further, (Behera & Chakraverty 2012; Behera & Chakraverty 2013a) solved complex fuzzy system of linear equations also.

We have used fuzzy finite element method (Chakraverty & Nayak 2012) to solve the uncertain heat conduction-convection problem. Here we have considered both Triangular Fuzzy Number (TFN) and Trapezoidal Fuzzy Number (TRFN) for imprecise parameters involved in the problem. The utility and application of the proposed method is discussed by considering a nonhomogeneous fin in conduction-convection system. The obtained result lies in an uncertain region. These uncertain solutions are discussed in detail along with its special cases.

5.1.1 Finite Element Formulation for Tapered Fin

Let us consider a tapered fin with plane surfaces on the top and bottom. The fin also loses heat to the ambient via the tip. The thickness of the fin varies linearly from d_1 at the base to d_2 at the tip. The width b remains constant throughout the fin and L is the length.

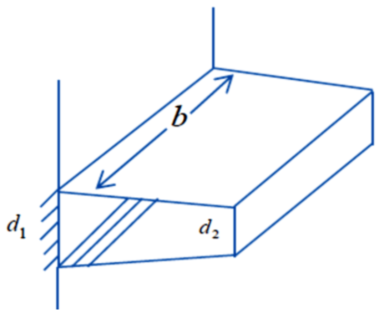

Figure 5.1. Model diagram of tapered fin

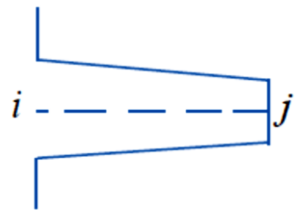

Figure 5.2. Tapered fin having two nodes

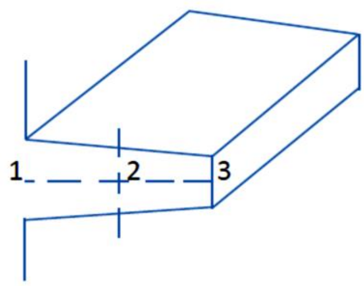

Figure 5.3. Two element discretization of Tapered fin

Here, a typical element e_1 is having nodes i and j respectively has been shown in Figure 5.1. The corresponding area (A_i and A_j) and perimeter (P_i and P_j) for nodes i and j respectively are

$$A_i = bd_i \,;\; A_j = bd_j \,;\; P_i = 2(b + d_i);\; P_j = 2(b + d_j) \tag{5.1}$$

The shape functions for each elements are $\left(1 - \frac{x}{L}\right)$ and $\frac{x}{L}$ and area of the fin varies linearly. So the area A of tapered fin may be expressed as follows.

$$\begin{aligned} A &= A_i\left(1 - \frac{x}{L}\right) + A_j\left(\frac{x}{L}\right) \\ &= A_i - \frac{A_i - A_j}{L}x \end{aligned} \tag{5.2}$$

Similarly, perimeter P of tapered fin may be written as

$$\begin{aligned} P &= P_i\left(1 - \frac{x}{L}\right) + P_j\left(\frac{x}{L}\right) \\ &= P_i - \frac{P_i - P_j}{L}x \end{aligned} \tag{5.3}$$

The stiffness matrix for the corresponding tapered fin (Lewis et al., 2004) is

$$[K] = \frac{k}{l}\left(\frac{A_i + A_j}{2}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hl}{12}\begin{bmatrix} 3P_i + P_j & P_i + P_j \\ P_i + P_j & P_i + 3P_j \end{bmatrix} \tag{5.4}$$

and the force vector is (Lewis et al. 2004)

$$\{f\} = \frac{Gl}{6}\begin{Bmatrix} 2A_i + A_j \\ A_i + 2A_j \end{Bmatrix} - \frac{ql}{6}\begin{Bmatrix} 2P_i + P_j \\ P_i + 2P_j \end{Bmatrix} + \frac{hT_a l}{6}\begin{Bmatrix} 2P_i + P_j \\ P_i + 2P_j \end{Bmatrix} + hT_a A\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \tag{5.5}$$

where, G is the heat source per unit volume, q is heat flux, h is heat transfer coefficient, k is thermal conductivity, T_a is ambient temperature and l is length of each element. Here in Eq. (5.5) the last term is valid only for the element at the end face with area A. For all other element the last term of the Eq. (5.5) is zero.

Next, considering finite element formulation of tapered fin, we have taken a numerical example of the same and obtained the corresponding results under uncertain environment.

5.1.2 Numerical Investigation

Consider a tapered fin where the thickness (d) varies from the base $2mm$ to the tip $1mm$ (Figure 5.1). The base temperature is maintained at $100^\circ C$. The total length of the fin is $20mm$ and width (b) is $3mm$. Further, corresponding data for this problem is provided in Tables 5.1 and 5.2.

Table 5.1. Fuzzy parameters (Triangular Fuzzy Number)

Parameters	Notations	Crisp value	Fuzzy value
Heat transfer coefficient	h	$120 W/m^2\,{}^\circ C$	$[115, 120, 125] W/m^2\,{}^\circ C$
Thermal conductivity	k	$200 W/m\text{–}{}^\circ C$	$[195, 200, 205] W/m\text{–}{}^\circ C$
Ambient Temperature	T_a	$25^\circ C$	$[20, 25, 30]^\circ C$

Table 5.2. Fuzzy parameters (Trapezoidal Fuzzy Number)

Parameters	Notations	Crisp value	Fuzzy value
Heat transfer coefficient	h	$120 W/m^2\,{}^\circ C$	$[115, 119, 121, 125] W/m^2\,{}^\circ C$
Thermal conductivity	k	$200 W/m\text{–}{}^\circ C$	$[195, 199, 201, 205] W/m\text{–}{}^\circ C$
Ambient Temperature	T_a	$25^\circ C$	$[20, 24, 26, 30]^\circ C$

Initially the nodal temperatures of tapered fin under conduction-convection system is analysed for crisp values for the sake of completeness. The computed nodal temperatures for different number of discretizations of the same domain are given in Table 5.3. Corresponding nodal temperatures for different number of discretization have been presented graphically in Figure 5.4.

Now considering the imprecise values as fuzzy, mentioned above, we get a series of nodal temperatures depending upon the value of α. When the value of α becomes one (that is crisp) we get the series of nodal temperatures which are presented in Table 5.3. Similarly when the value of α is zero then the obtained results are mentioned in Table 5.4. It is worth mentioning that the value of $\alpha = 0$ gives the interval results.

Table 5.3. Nodal temperatures of tapered fin (crisp value)

Elements → Temperatures (in $^{\circ}C$) ↓	2	4	8	16
T_1	100	100	100	100
T_2	**90.2798**	94.59511	97.16331	98.54563
T_3	86.22617	**90.40015**	94.61663	97.16265
T_4		87.59145	92.37163	95.85208
T_5		86.53492	**90.44605**	94.61518
T_6			88.86572	93.45358
T_7			87.66768	92.36925
T_8			86.89835	91.36401
T_9			86.64014	**90.44135**
T_{10}				89.60477
T_{11}				88.85846
T_{12}				88.20742
T_{13}				87.65763
T_{14}				87.21625
T_{15}				86.89188
T_{16}				86.69491
T_{17}				86.63797

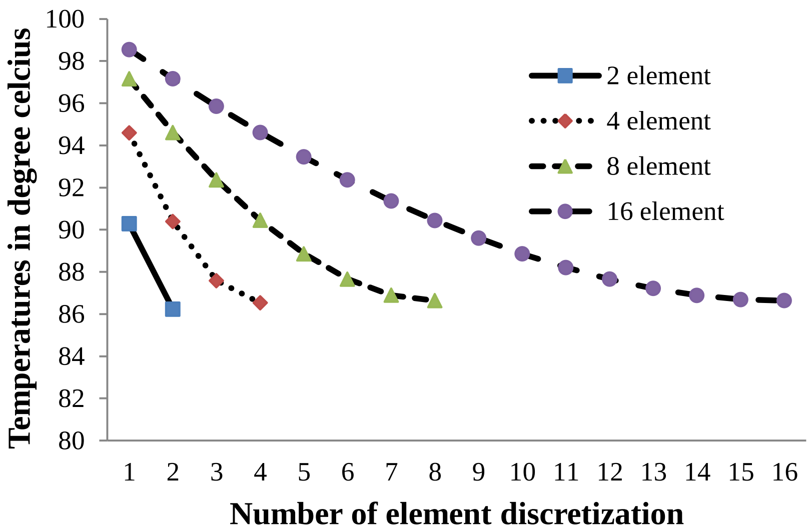

Figure 5.4. Nodal temperatures of tapered fin (crisp values)

Table 5.4. Nodal temperatures of tapered fin (Triangular fuzzy values)

Elements → / Temperatures (in $^{\circ}C$) ↓		2	4	8	16
T_1	Left	100	100	100	100
	Right	100	100	100	100
T_2	Left	**89.7674**	94.31276	97.01493	98.47007
	Right	**90.81915**	94.89252	97.31966	98.62527
T_3	Left	85.47571	**89.89299**	94.33342	97.01486
	Right	87.01505	**90.93384**	94.91488	97.31839
T_4	Left		86.92682	91.96786	95.63545
	Right		88.29033	92.79662	96.08028
T_5	Left		85.79875	**89.937**	94.3332
	Right		87.30858	**90.9816**	94.91216
T_6	Left			88.26804	93.10981
	Right			89.49425	93.81553
T_7	Left			86.99991	91.9674
	Right			88.36966	92.79225
T_8	Left			86.18306	90.90804
	Right			87.65007	91.84387
T_9	Left			85.90237	**89.93525**
	Right			87.41529	**90.97385**
T_{10}	Left				89.05274
	Right				90.1855
T_{11}	Left				88.26489
	Right				89.48277
T_{12}	Left				87.57698
	Right				88.8704
T_{13}	Left				86.99527
	Right				88.35404
T_{14}	Left				86.5273
	Right				87.9405
T_{15}	Left				86.18209
	Right				87.63794
T_{16}	Left				85.97054
	Right				87.4562
T_{17}	Left				85.90588
	Right				87.40727

Corresponding nodal temperatures of Table 5.4 for different number of discretizations with fuzzy parameters are also shown in Figure 5.5.

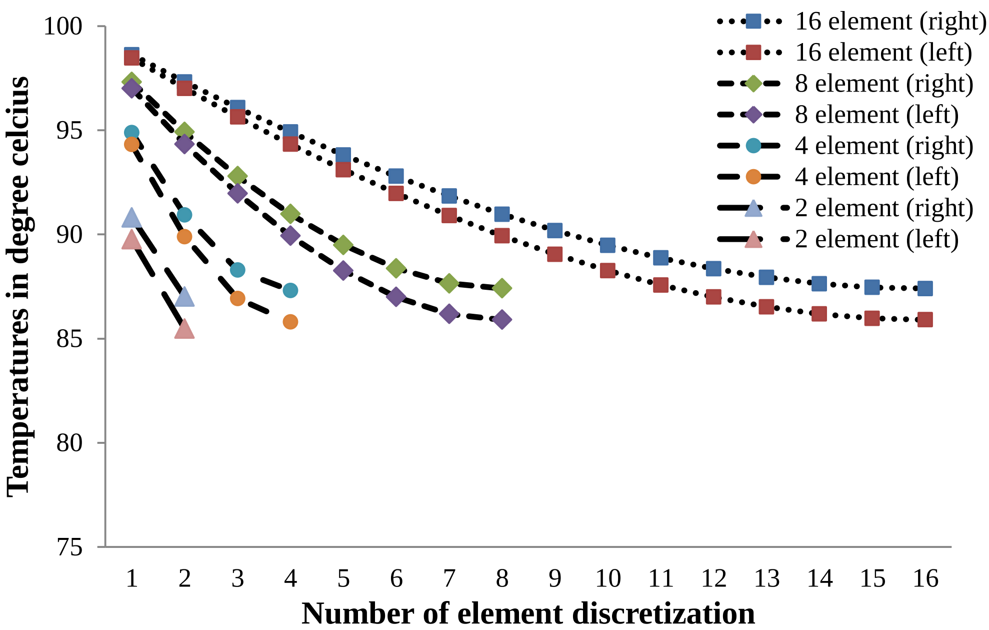

Figure 5.5. Nodal temperatures of tapered fin (left and right values)

Here the base of the tapered fin is maintained at a constant temperature of $100^{\circ}C$. Width of the fin is varying linearly so the surface of contact changes with the ambient temperatures. Now considering the above criteria, the uncertain temperatures are obtained when the system is at equilibrium. Firstly the crisp parameters are considered to study the variation of temperature for a tapered fin. The domain is discretized into various numbers of elements and the nodal temperatures are obtained. It is observed that the temperature decreases as we move away from the base of the fin. To observe the above variation of temperature through the domain we have considered 2, 4, 8 and 16 element discretization. The resultant temperatures (crisp) for various number of discretization are presented in Table 5.3. In view of Table 5.3 let us fix the node which divides the domain into two equal parts for different number of discretization and we call it as central node. The corresponding temperatures of the central node occur at T_2, T_3, T_5 and T_9 for 2, 4, 8 and 16 elements discretization respectively. It is observed that the central nodal temperature converges as we vary the number of discretization. Figure 5.4 shows that the distribution graph of temperatures become steeper with more number of discretization of the considered tapered fin. So, one may get the better distribution of temperature along the domain by considering more number of discretized elements for the same domain, as expected. The above is well known although to investigate the corresponding uncertain results we do study the usual case of the crisp problem.

As mentioned earlier, in actual practice the involved parameters may not be crisp, rather it may be uncertain. To handle such uncertainties here the parameters are taken as interval/fuzzy which are given in Tables 5.1 (TFN) and 5.2 (TRFN) respectively. The fuzzy values are converted into α -cut form. These values give a piecewise normalised function. If the normality is considered then the temperature will be equivalent to the temperature for crisp form. So the results varies over the value of α which will give the results in term of fuzzy numbers.

The variation of temperatures using the data of Table 5.1 (TFN) is given in Table 5.4. It may be noted that the left and right values of the resultant uncertain temperatures are given in Table 5.4. For better visualization of the values these are plotted in Figure 5.5. The widths of the uncertain results are studied next. We get a narrow width for various temperatures in different discretization and it maintains a regular pattern. Results are compared by considering the central node of the tapered fin. Corresponding triangular fuzzy temperatures at central node are also presented in Figures 5.6 to 5.9.

Further trapezoidal fuzzy values are taken for uncertain parameters involved in the system. Using proposed fuzzy finite element method the nodal temperatures at central node are shown in Figure 5.10. It may be seen from Figure 5.10 that the left and right temperatures for different discretization converge which shows the efficacy of the proposed method.

It may be a point to be noted that the reliability of the fuzzy results can be seen in the special cases viz. crisp and interval which are derived from the fuzzy values. As such three cases are reported with respect to the above.

Case-1

Here we have considered only left monotonic increasing functions of the resultant temperatures. The resulting temperatures vary with the value of membership functions. Assigning zero for the value of α we get the left bound of the uncertain fuzzy temperatures. Similarly if the value of α is taken as one then we get right bound of the left monotonic increasing functions which are the centre value of TFN.

Case-2

In this case only right monotonic decreasing functions of the resultant temperatures are considered. Resulting temperatures vary with the value of membership functions. Assigning zero for the value of α we get the right bound of the uncertain fuzzy temperatures. Similarly if the value of α is taken as one then we get left bound of the right monotonic decreasing functions which are the centre value of TFN.

Case-3

Now let us consider the case where the value of α is one for both the left and right monotonic functions. We observe that the resultant temperatures become same for both monotonic functions. If we consider TFN then we find that this is nothing but the centre value. For TRFN we get two different values for both monotonic functions. Here we get an interval of temperatures where the membership functions are normalised.

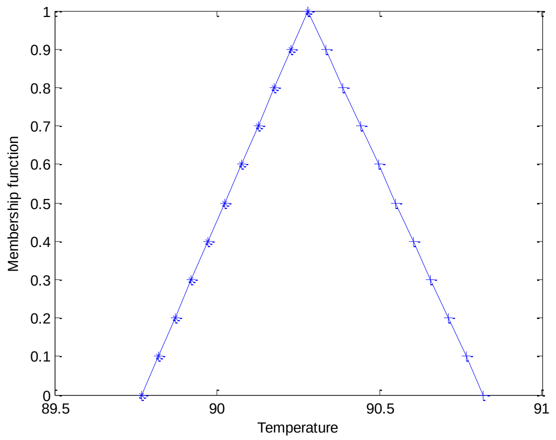

Figure 5.6. 2 element discretization

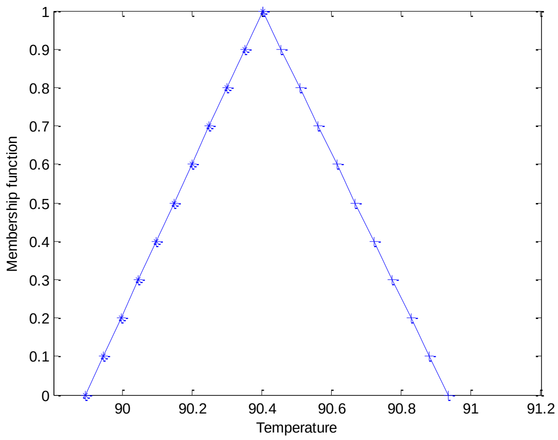

Figure 5.7. 4 element discretization

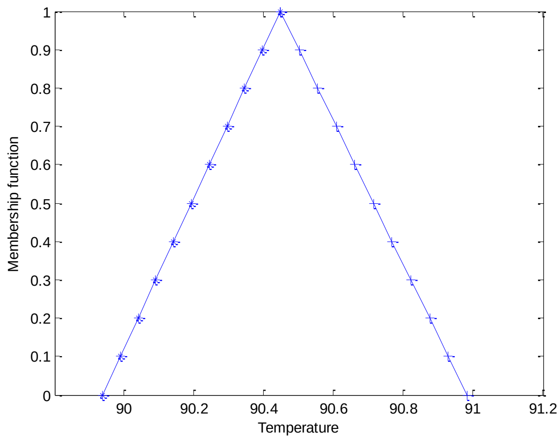

Figure 5.8. 8 element discretization

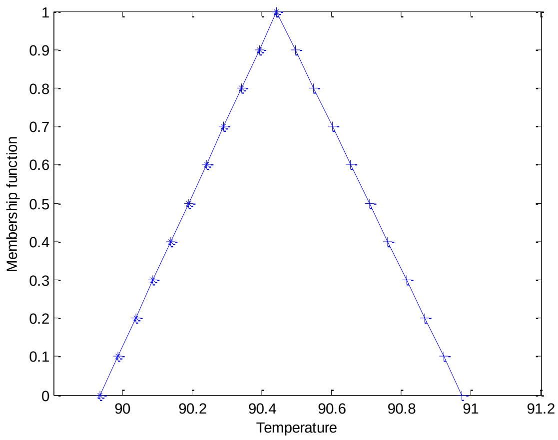

Figure 5.9. 16 element discretization

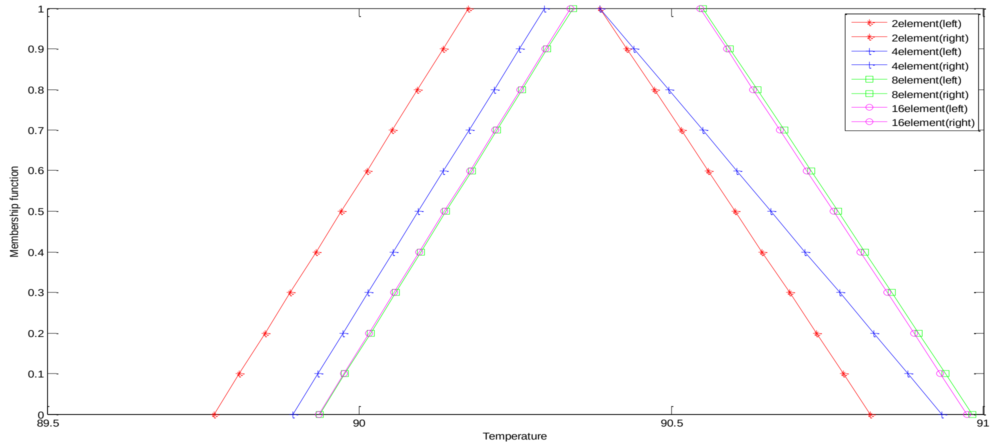

Figure 5.10. Central nodal temperatures (TRFN) for different element discretization

5.2 Conjugate Heat Transfer in Plate

Conjugate heat transfer refers to a heat transfer process involving an interaction of conduction within a solid body and the convection from the solid surface to fluid moving over the surface. Therefore, a realistic analysis of conjugate heat transfer problems necessitates the coupling of the conduction in the solid and the convection in the fluid. In view of this we may characterize the conjugate heat transfer in a plate as (i) when both plate and the surrounded fluid are at rest, (ii) when the plate is moving and surrounded fluid is at rest, (iii) when the surrounded fluid is moving and plate is at rest and finally (iv) when both the surrounded fluid and plate are moving. Here we have considered only the steady state case. As a result both the plate and fluid is at rest and the uncertain temperature is obtained under this environment.

Here fuzzy finite element method (Chapter 3) has been presented which may handle the uncertainty involved in heat transfer problems. As mentioned earlier, in this method the fuzzy parameters are converted into intervals which are then transformed into crisp form using a proposed transformation (Chakraverty & Nayak, 2012; Chakraverty & Nayak, 2013). Crisp representations of intervals are defined by symbolic parameterization. Traditional interval arithmetic is modified using the crisp representation of intervals. Further the detail of this method is presented in the subsequent sections. Using this method conjugate heat transfer in a square plate is studied and the quantified uncertain temperatures of the plate are investigated. Finally sensitivity of the uncertain parameters are analysed.

5.2.1 Formulation of the Problem

The real world problems depend on experimental observations and operating conditions which are influenced by the systems. In general, these problems contain uncertainty due to the involved parameters, experimental observations and operating conditions of the systems. In actual practice these uncertainties cannot be avoided. As mentioned earlier that one may handle this uncertainty by probabilistic method but due to the lack of knowledge about the distribution of random values for uncertain parameters we have considered these uncertain parameters as fuzzy. However it may not be easy to handle such uncertain data as fuzzy due to complicated fuzzy arithmetic. To tackle these problems we have taken Triangular Fuzzy Number (TFN) for the uncertain parameters and proposed fuzzy finite element method.

For the sake of completeness initially we have taken crisp parameters for the mentioned problem. The standard form of the finite element equations for a plate may be represented as

$$[K]\{T\}=\{f\} \tag{5.6}$$

where

$$[K]=\int_{\Omega}[B]^T[D][B]d\Omega+\int_{\Gamma}h[N]^T[N]d\Gamma \tag{5.7}$$

and

$$\{f\}=\int_{\Omega}G[N]^T d\Omega-\int_{\Gamma}q[N]^T d\Gamma+\int_{\Gamma}hT_a[N]^T d\Gamma \tag{5.8}$$

Here, h is convective heat transfer coefficient, q is heat input rate, T_a is ambient temperature, k is thermal conductivity of the material, $[K]$ is stiffness matrix and $\{f\}$ is the force vector.

Let us consider a typical element (Figure 5.11) of the domain and the temperature distribution may be written as

$$T=N_iT_i+N_jT_j+N_kT_k \tag{5.9}$$

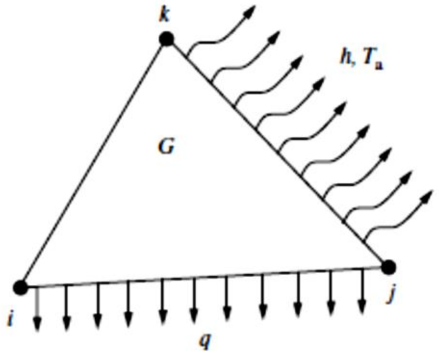

Figure 5.11. Typical two-dimensional triangular element with heat generation, heat flux and convection boundaries

The gradient matrix is given by

$$\{g\} = \begin{Bmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial N_i}{\partial x} & \dfrac{\partial N_j}{\partial x} & \dfrac{\partial N_k}{\partial x} \\ \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_j}{\partial y} & \dfrac{\partial N_k}{\partial y} \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix} = [B]\{T\} \tag{5.10}$$

where

$$[B] = \begin{bmatrix} \dfrac{\partial N_i}{\partial x} & \dfrac{\partial N_j}{\partial x} & \dfrac{\partial N_k}{\partial x} \\ \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_j}{\partial y} & \dfrac{\partial N_k}{\partial y} \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} \tag{5.11}$$

$$[D] = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \tag{5.12}$$

The elemental stiffness matrix is then written as

$$[K]^{(e)} = \frac{t}{4A} \left\{ k_x \begin{bmatrix} a_i^2 & a_i a_j & a_i a_k \\ a_i a_j & a_j^2 & a_j a_k \\ a_i a_k & a_j a_k & a_k^2 \end{bmatrix} + k_y \begin{bmatrix} b_i^2 & b_i b_j & b_i b_k \\ b_i b_j & b_j^2 & b_j b_k \\ b_i b_k & b_j b_k & b_k^2 \end{bmatrix} \right\} + \frac{htl_{jk}}{6} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \tag{5.13}$$

It should be noted that in the above equation, $d\Omega$ is equal to tdA and $d\Gamma$ is equal to tdl, where t is the thickness of the plate and l is the length of an element side on the domain boundary. Similarly the forcing vector may be written as

$$\{f\}^{(e)} = \frac{GAt}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} - \frac{qtl_{ij}}{2} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} + \frac{hT_a tl_{jk}}{2} \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} \tag{5.14}$$

If we need to have a point source G^* instead of a uniform source G then the first term of the Eq. (5.14) is replaced by

$$\{f\} = G^* t \begin{Bmatrix} N_i \\ N_j \\ N_k \end{Bmatrix}_{(x_0, y_0)} \tag{5.15}$$

It may now be worth mentioning that in general when heat flows through domain, uncertainty occurs due to the imprecise value of operating parameters viz. geometry, diffusion coefficients and thermal conductivity etc. Here these uncertain parameters are taken as fuzzy. To investigate the uncertain temperature distribution we formulate fuzzy finite element method with linear triangular fuzzy element discretising the domain. Next, the proposed fuzzy finite element procedure for triangular elements is discussed.

Let us consider that the coordinates of linear triangular elements are fuzzy and hence we may write

$$\begin{aligned} \tilde{x} &= \tilde{N}_1 \tilde{x}_1 + \tilde{N}_2 \tilde{x}_2 + \tilde{N}_3 \tilde{x}_3; \\ \tilde{y} &= \tilde{N}_1 \tilde{y}_1 + \tilde{N}_2 \tilde{y}_2 + \tilde{N}_3 \tilde{y}_3; \\ \tilde{N} &= \tilde{N}_1 + \tilde{N}_2 + \tilde{N}_3; \end{aligned} \tag{5.16}$$

where $\tilde{N}_i \ (i = 1, 2, 3)$ are nondimensionalized coordinates and '~' is used for fuzzy numbers or fuzzy sets.

The above Eq. (5.16) in matrix form may be represented as

$$\begin{bmatrix} 1 & 1 & 1 \\ \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \\ \tilde{y}_1 & \tilde{y}_2 & \tilde{y}_3 \end{bmatrix} \begin{Bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \\ \tilde{N}_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ \tilde{x} \\ \tilde{y} \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \\ \tilde{N}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \\ \tilde{y}_1 & \tilde{y}_2 & \tilde{y}_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ \tilde{x} \\ \tilde{y} \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \\ \tilde{N}_3 \end{Bmatrix} = \begin{bmatrix} \tilde{x}_2 \tilde{y}_3 - \tilde{x}_3 \tilde{y}_2 & \tilde{y}_2 - \tilde{y}_3 & \tilde{x}_3 - \tilde{x}_2 \\ \tilde{x}_3 \tilde{y}_1 - \tilde{x}_1 \tilde{y}_3 & \tilde{y}_3 - \tilde{y}_1 & \tilde{x}_1 - \tilde{x}_3 \\ \tilde{x}_1 \tilde{y}_2 - \tilde{x}_2 \tilde{y}_1 & \tilde{y}_1 - \tilde{y}_2 & \tilde{x}_2 - \tilde{x}_1 \end{bmatrix} \begin{Bmatrix} 1 \\ \tilde{x} \\ \tilde{y} \end{Bmatrix}$$

where area of the fuzzy triangle is $\tilde{\Delta} = \dfrac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \\ \tilde{y}_1 & \tilde{y}_2 & \tilde{y}_3 \end{bmatrix}$.

Now let us denote

$$
\begin{aligned}
&\tilde{a}_1 = \tilde{x}_3 - \tilde{x}_2,\ \tilde{a}_2 = \tilde{x}_1 - \tilde{x}_3,\ \tilde{a}_3 = \tilde{x}_2 - \tilde{x}_1;\\
&\tilde{b}_1 = \tilde{y}_2 - \tilde{y}_3,\ \tilde{b}_2 = \tilde{y}_3 - \tilde{y}_1,\ \tilde{b}_3 = \tilde{y}_1 - \tilde{y}_2;\\
&\tilde{c}_1 = \tilde{x}_2\tilde{y}_3 - \tilde{x}_3\tilde{y}_2,\ \tilde{c}_2 = \tilde{x}_3\tilde{y}_1 - \tilde{x}_1\tilde{y}_3,\ \tilde{c}_3 = \tilde{x}_1\tilde{y}_2 - \tilde{x}_2\tilde{y}_1.
\end{aligned} \tag{5.17}
$$

If $\tilde{T}$ is the flux distribution then it may be written as

$$\tilde{T} = \tilde{N}_1\tilde{T}_1 + \tilde{N}_2\tilde{T}_2 + \tilde{N}_3\tilde{T}_3 \tag{5.18}$$

The differentiation and integration formulae are then given by,

$$\frac{\partial}{\partial\tilde{x}} = \sum_{i=1}^{3}\frac{\tilde{b}_i}{2\tilde{\Delta}}\frac{\partial}{\partial\tilde{N}_i}, \quad \frac{\partial}{\partial\tilde{y}} = \sum_{i=1}^{3}\frac{\tilde{a}_i}{2\tilde{\Delta}}\frac{\partial}{\partial\tilde{N}_i} \tag{5.19}$$

and

$$\iint_R \tilde{N}_1^p\tilde{N}_2^q\tilde{N}_3^r\, d\tilde{\Delta} = \frac{p!q!r!}{(p+q+r+2)!}(2\tilde{\Delta}) \tag{5.20}$$

Hence

$$\frac{\partial\tilde{T}^{(e)}}{\partial\tilde{x}} = \frac{1}{2\tilde{\Delta}}\left\{\tilde{b}_1\tilde{T}_1 + \tilde{b}_2\tilde{T}_2 + \tilde{b}_3\tilde{T}_3\right\} = \begin{bmatrix} \frac{\tilde{b}_1}{2\tilde{\Delta}} & \frac{\tilde{b}_2}{2\tilde{\Delta}} & \frac{\tilde{b}_3}{2\tilde{\Delta}} \end{bmatrix}\begin{Bmatrix} \tilde{T}_1 \\ \tilde{T}_2 \\ \tilde{T}_3 \end{Bmatrix} \tag{5.21}$$

Similarly,

$$\frac{\partial\tilde{T}^{(e)}}{\partial\tilde{y}} = \frac{1}{2\tilde{\Delta}}\left\{\tilde{a}_1\tilde{T}_1 + \tilde{a}_2\tilde{T}_2 + \tilde{a}_3\tilde{T}_3\right\} = \begin{bmatrix} \frac{\tilde{a}_1}{2\tilde{\Delta}} & \frac{\tilde{a}_2}{2\tilde{\Delta}} & \frac{\tilde{a}_3}{2\tilde{\Delta}} \end{bmatrix}\begin{Bmatrix} \tilde{T}_1 \\ \tilde{T}_2 \\ \tilde{T}_3 \end{Bmatrix} \tag{5.22}$$

Here the validity of the integration and differentiation terms may easily be proved by using the arithmetic (Chakraverty & Nayak 2012).

Using the above formulation, one may get the stiffness matrices and those are written for each element as

$$\left[\tilde{K}\right]^{(e)} = \frac{\tilde{t}}{4\tilde{\Delta}}\left\{\tilde{k}_x\begin{bmatrix} \tilde{a}_1^{\,2} & \tilde{a}_1\tilde{a}_2 & \tilde{a}_1\tilde{a}_{33} \\ \tilde{a}_1\tilde{a}_{22} & \tilde{a}_2^{\,2} & \tilde{a}_2\tilde{a}_{33} \\ \tilde{a}_1\tilde{a}_{33} & \tilde{a}_2\tilde{a}_3 & \tilde{a}_3^{\,2} \end{bmatrix} + \tilde{k}_y\begin{bmatrix} \tilde{b}_1^{\,2} & \tilde{b}_1\tilde{b}_2 & \tilde{b}_1\tilde{b}_3 \\ \tilde{b}_1\tilde{b}_2 & \tilde{b}_2^{\,2} & \tilde{b}_2\tilde{b}_3 \\ \tilde{b}_1\tilde{b}_3 & \tilde{b}_2\tilde{b}_3 & \tilde{b}_3^{\,2} \end{bmatrix}\right\} + \frac{\tilde{h}\tilde{t}\tilde{l}_{jk}}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \tag{5.23}$$

where $[\tilde{K}]$ is the fuzzy stiffness matrix.

Next the force vector $\{\tilde{f}\}$ for each element may be written as

$$\left\{\tilde{f}\right\}^{(e)} = \frac{\tilde{G}\tilde{A}\tilde{t}}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} - \frac{\tilde{q}\tilde{t}\tilde{l}_{ij}}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} + \frac{\tilde{h}\tilde{T}_a\tilde{t}\tilde{l}_{jk}}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} \tag{5.24}$$

Now the above formulations may be used for proposed fuzzy finite element method and finally we get the following algebraic equations in matrix form

$$[\tilde{K}]\{\tilde{T}\} = \{\tilde{f}\} \tag{5.25}$$

5.2.2 Example Problem and Results

The above formulations are used for a square plate of size $5cm \times 5cm$. A constant heat flux is specified to the left wall of the plate and the right side of the wall is maintained at a constant temperature of $100^{\circ}C$. The bottom wall is insulated and there is convective heat transfer along the top wall. Initially plate temperatures are investigated for crisp parameters by using traditional finite element method. Next the parameters are taken as TFN and the uncertain temperatures are obtained. Obtained results are analysed and the sensitiveness of uncertain fuzzy parameters are studied in detail. The uncertain temperatures for plate is analysed for different cases. In Table 5.5 we have given the values of the parameters considered. Model diagram of a square plate with boundary conditions are given in Figure 5.12. This plate is discretized into 18, 32 and 98 elements, which are shown in Figures 5.12 to 5.14.

Table 5.5. Crisp and fuzzy values of involved parameters

Parameters	Crisp value	TFN
h	$1.2 W/cm^2K$	$[1, 1.2, 1.5] W/cm^2K$
q	$2W/cm^2$	$[1.5, 2, 2.5] W/cm^2$
T_a	$30^{\circ}C$	$[25, 30, 35]^{\circ}C$
k	$200 W/m{-}^{\circ}C$	$[190, 200, 210] W/m{-}^{\circ}C$

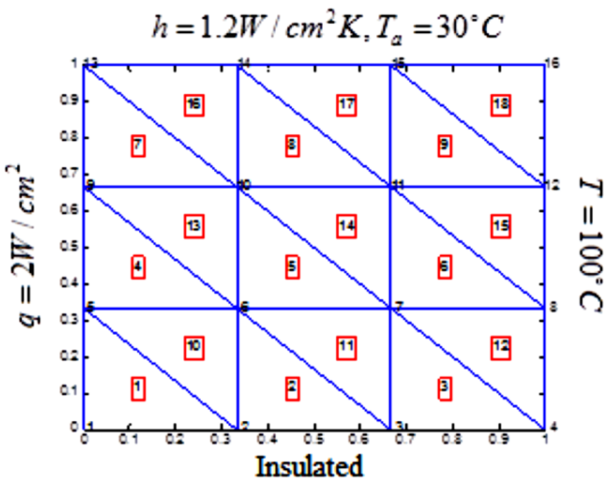

Figure 5.12. Model diagram of a plate having 18 element discretizations with boundary conditions

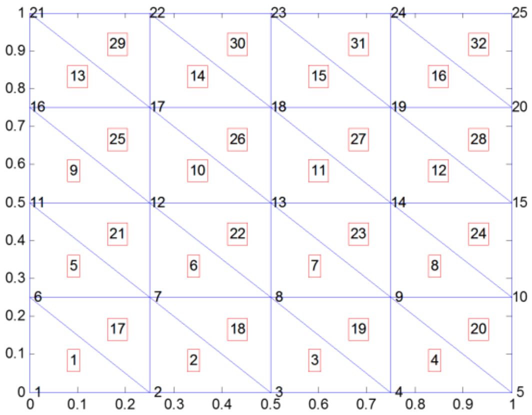

Figure 5.13. 32 element discretization of the plate

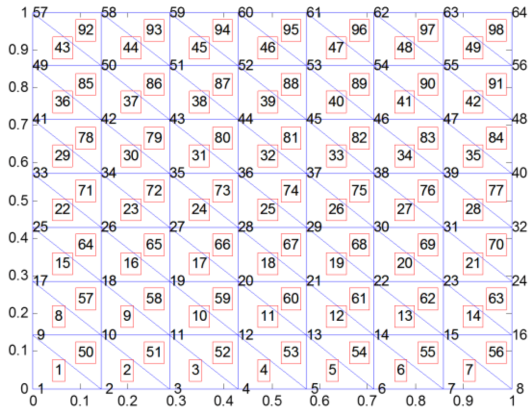

Figure 5.14. 98 element discretization of the plate

In view of these uncertain crisp and fuzzy values, we have demonstrated the proposed method for the said problem. The temperatures for 18 element discretizations of the plate for various boundary conditions are depicted in Table 5.6.

Table 5.6. Comparison of crisp (Lewis et al. 2004) with centre value of fuzzy temperatures

Fuzzy parameters	Crisp Temperatures (FEM)	Fuzzy Temperatures (FFEM)
q	49.39465	[48.87894, 49.39465, 49.91036]
q and k	49.39465	[48.15113, 49.39465, 50.5993]
h and T_a	49.39465	[46.61134, 49.39465, 51.7083]
h, T_a, k	49.39465	[48.0931, 49.39465, 50.55269]
h, q, T_a	49.39465	[48.36927, 49.39465, 50.44243]
q, T_a, k	49.39465	[44.33439, 49.39465, 54.3078]
h, q, k	49.39465	[46.61134, 49.39465, 51.12802]
h, q, k, T_a	49.39465	[47.51282, 49.39465, 51.00026]

The method as discussed previously has been used to handle the uncertain parameters for this problem and the variations of temperatures are encrypted in the following Figures 5.15 to 5.30. Different combinations of the uncertain parameters or hybrid boundary conditions are considered and the distributions of uncertain fuzzy temperatures are depicted in Figures 5.15 to 5.30.

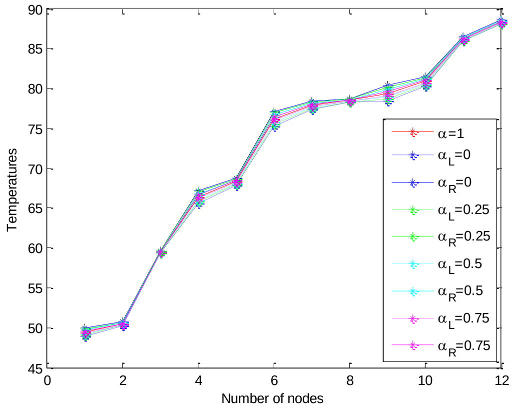

Figure 5.15. Only q is fuzzy

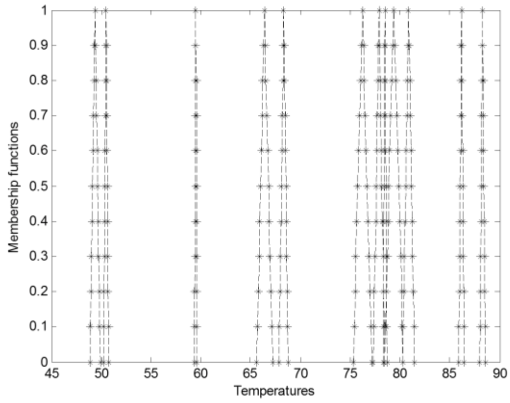

Figure 5.16. TFN plot when only q is fuzzy

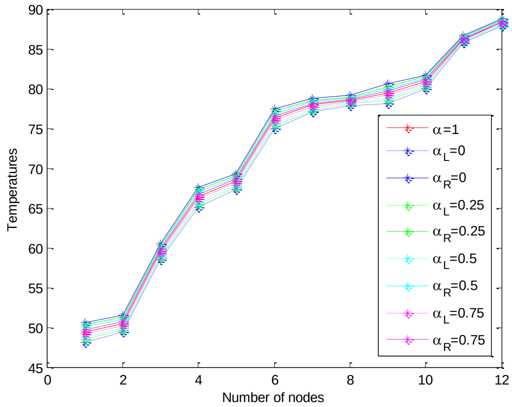

Figure 5.17. Only q and k are fuzzy

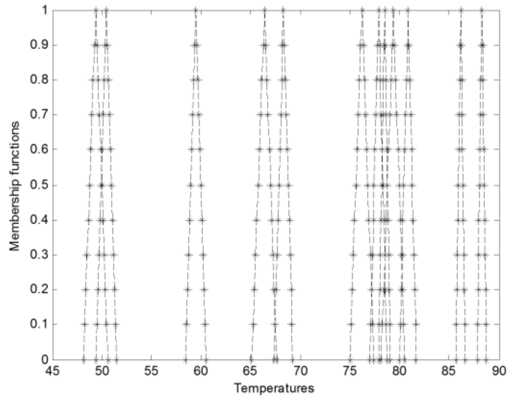

Figure 5.18. TFN plot when only q and k are fuzzy

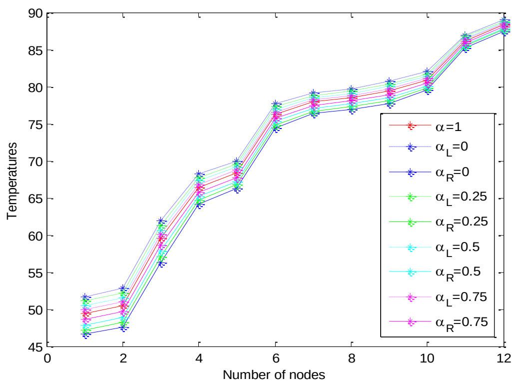

Figure 5.19. Only h and T_a are fuzzy

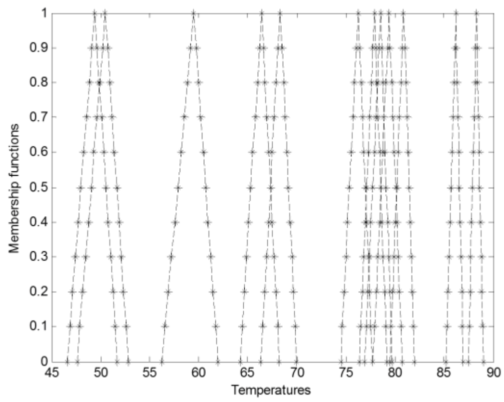

Figure 5.20. TFN plot when only h and T_a are fuzzy

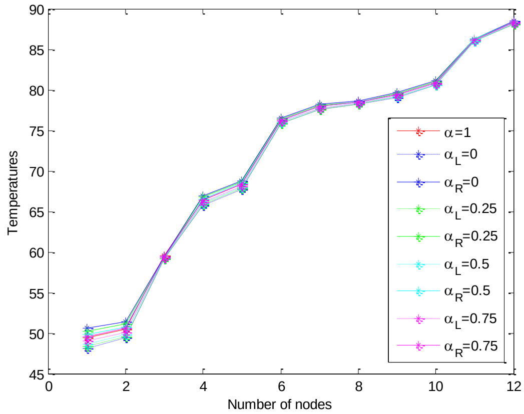

Figure 5.21. h, T_a, k are fuzzy

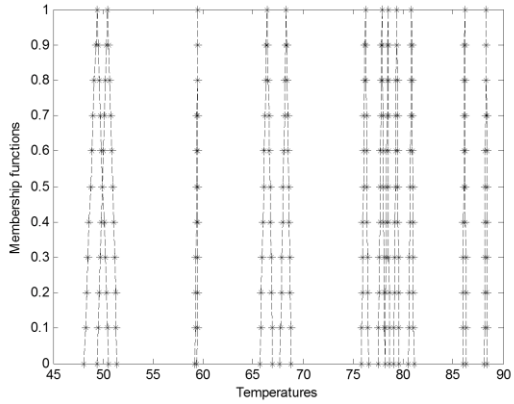

Figure 5.22. TFN plot when h, T_a, k are fuzzy

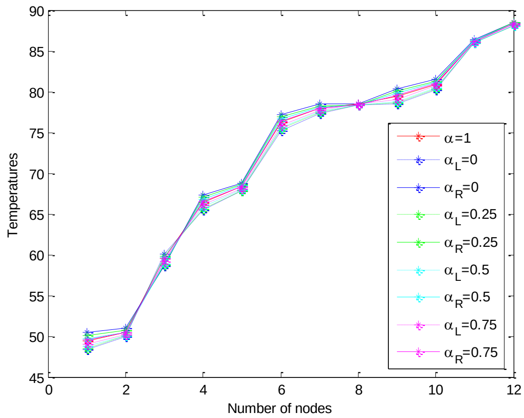

Figure 5.23. h, q, T_a are fuzzy

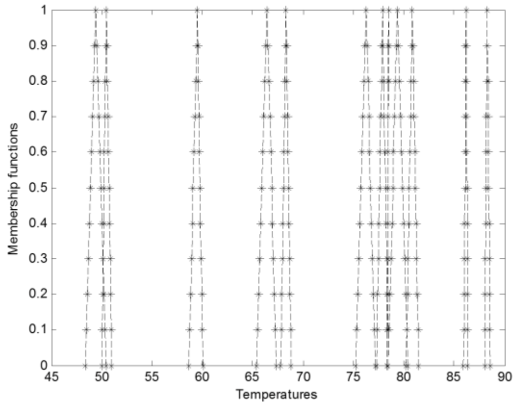

Figure 5.24. TFN plot when h, q, T_a are fuzzy

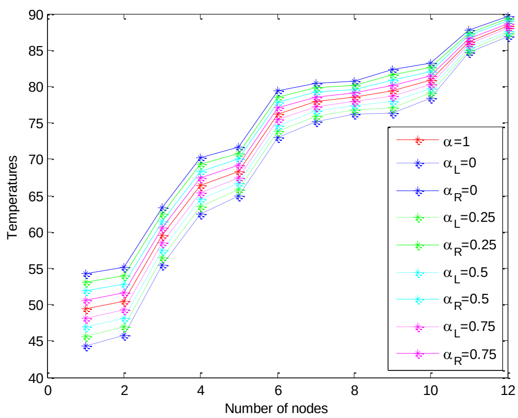

Figure 5.25. q, T_a, k are fuzzy

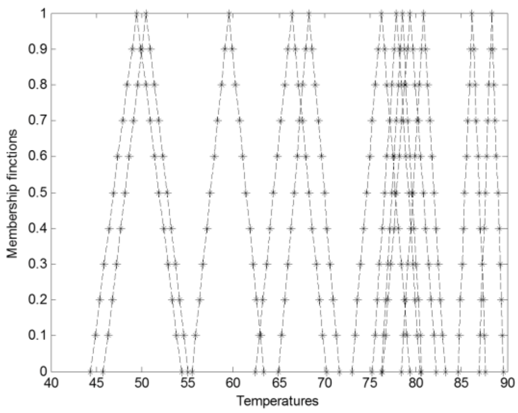

Figure 5.26. TFN plot when q, T_a, k are fuzzy

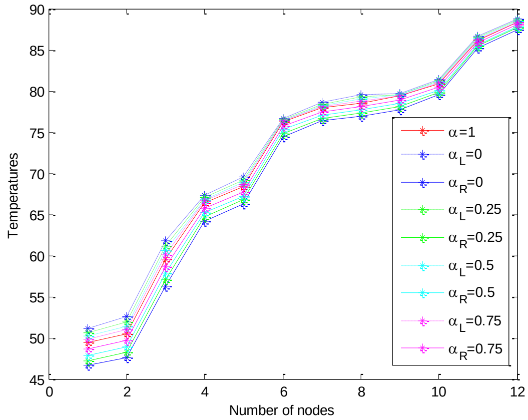

Figure 5.27. h, q, k are fuzzy

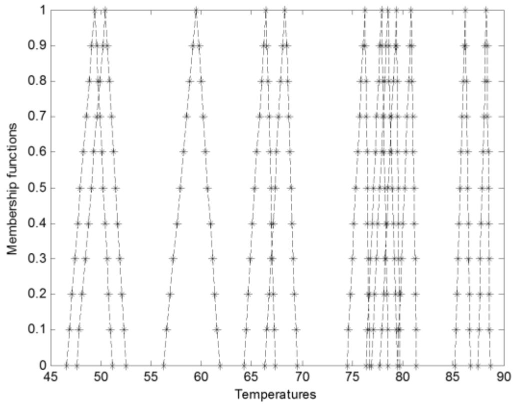

Figure 5.28. TFN plot when h, q, k are fuzzy

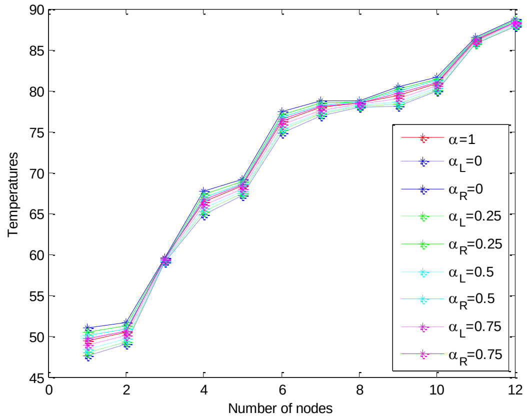

Figure 5.29. All parameters are fuzzy

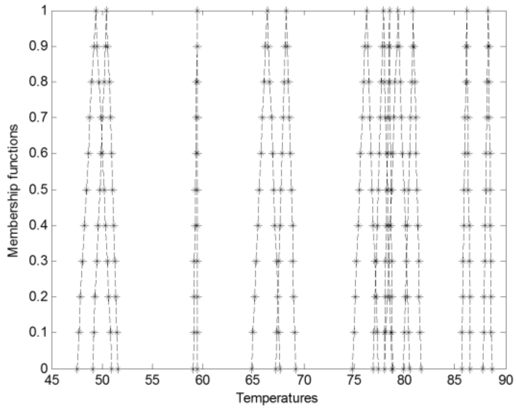

Figure 5.30. Nodal temperatures when all parameters are fuzzy

Here we have considered different combinations of fuzzy parameters and investigated the variation of uncertain temperatures. These are discussed in the following cases.

Case 1

In this case we have considered the left boundary condition as fuzzy. First the heat input rate and then both the heat input rate and thermal conductivity are taken as fuzzy. When we take only heat input rate as fuzzy, the obtained fuzzy temperatures for different membership functions are shown in Figure 5.15 and the corresponding TFN plot is shown in Figure 5.16. Similarly when both the heat input rate and thermal conductivity are taken as fuzzy, the obtained fuzzy temperatures for different membership functions are depicted in Figure 5.17 and the corresponding TFN plot is shown in Figure 5.18.

Case 2

The top wall of the plate is supplied to a convective heat transfer. For the boundary conditions of top wall initially we have taken both the convective heat transfer coefficient and ambient temperatures as fuzzy then thermal conductivity is also considered as fuzzy along with the above two. The investigated results for both the fuzzy convective heat transfer coefficient and ambient temperatures are depicted in Figures 5.19 and 5.20. In Figure 5.19, the plot for uncertain temperatures are presented for different membership functions and in Figure 5.20, the obtained TFN values are shown. Similarly the results are cited in Figures 5.21 and 5.22 when the values of heat transfer coefficient, thermal conductivity and ambient temperatures are taken as fuzzy.

Case 3

In this case, parameters involved in both the left and top boundary of the square plate are considered as fuzzy. Solving this problem we get distribution of uncertain fuzzy temperatures which are shown in Figures 5.23 and 5.24. For different membership functions the distribution of uncertain temperatures are plotted in Figure 5.23 and corresponding TFN plot is given in Figure 5.24.

Case 4

We have considered only one parameter as crisp and other three parameters are fuzzy except the combinations of parameters discussed in case 3. The distribution of observed uncertain temperatures for square plate is shown in Figures 5.25 and 5.26 where heat input rate, ambient temperatures and thermal conductivity are all taken as fuzzy. Whereas, considering fuzzy values of convective heat transfer coefficient, heat input rate and thermal conductivity, the results are presented in Figures 5.27 and 5.28.

Case 5

When all four parameters (heat input rate, convective heat transfer coefficient, ambient temperatures and thermal conductivity) are taken as fuzzy then the computed results are depicted in Figure 5.29 for different membership functions whereas the plot for TFN is given in Figure 5.30.

We have considered imprecise parameters to quantify the uncertain temperatures of the plate. Obtained fuzzy temperatures may easily be analysed from Figures 5.15 to 5.30. The distribution of uncertain temperatures are plotted in Figures 5.15, 5.17, 5.19, 5.21 5.23, 5.25, 5.27 and 5.29 for few particular values of alpha level sets. It may be seen that if the value of alpha increases from zero to one, we get a narrow width of the temperatures and uncertainty decreases gradually which finally gives crisp results for $\alpha = 1$. For example if we consider Figure 5.15, it may be easily noticed that the width of the uncertain temperature varies with the variation of membership functions. For different values of alpha ($\alpha = 1, 0.25, 0.5, 0.75$), the temperatures are plotted. Here α_L and α_R represents the left and right bounds of the uncertain temperatures. In the same manner we may get different distribution of temperatures by applying different values for the membership functions.

The sensitiveness of the uncertain/fuzzy parameters is easily studied by considering the different cases discussed above. One can see the variations of uncertain width of the temperature, from the plots presented in various cases. We see that when the boundary condition for left edge is considered as fuzzy then Figure 5.17 gives larger width of uncertain temperatures as compared to Figure 5.15 for different level of α-cut fuzzy sets. Similarly it is seen that the deviation of uncertain temperature from the centre value of each TFN is much larger in Figure 5.18 than Figure 5.16. So it is concluded that when left boundary is taken as fuzzy, the thermal conductivity is more sensitive as it maximizes the possibility of uncertain temperatures. But in the case when top boundary is taken as fuzzy, the thermal conductivity minimizes the possibility of uncertain temperatures which may be seen from Figures 5.21 and 5.22. Here addition of thermal conductivity as fuzzy with the fuzzy value of convective heat transfer and ambient temperature give smaller width of uncertain temperatures which may be understood from the obtained results given in Figures 5.19 and 5.20. Further, it is observed that the uncertainty in temperatures decreases drastically if we consider case 3. When all four parameters viz. heat input rate, convective heat transfer coefficient, ambient temperatures and thermal conductivity are taken as fuzzy then we get comparatively less width of uncertain temperatures which are presented in Figures 5.29 and 5.30. Finally it may be noticed that when width of the uncertain temperatures are considered, we get best possibility of distribution of uncertain temperatures for the case when thermal conductivity, convective heat transfer and ambient temperatures are taken as fuzzy.

The validity and efficacy of the proposed method may be seen by the convergence of the uncertain plate temperatures for various discretizations. These may be observed from Table 5.6. The investigated TFN values are pictorially represented in Figures 5.31 and 5.32. It is seen that the TFN values for both the minimum and maximum nodal temperatures of the plate converges if we consider more number of discretizations.

Table 5.7. TFN values of minimum and maximum uncertain nodal temperatures for various element discretizations of the plate

Number of Elements	Minimum Nodal TFN Temperatures	Maximum Nodal TFN Temperatures
2	[41.1725, 42.1428, 42.4846]	[72.5599, 73.5714, 74.2185]
8	[46.6396, 48.4751, 50.0416]	[82.3680, 83.0585, 83.5835]
18	[47.5128, 49.3946, 51.0002]	[87.8165, 88.3055, 88.6874]
32	[47.8355, 49.7225, 51.3298]	[90.7927, 91.1689, 91.4669]
50	[47.9851, 49.8718, 51.2748]	[92.6186, 92.9234, 93.1669]
72	[48.0647, 49.8264, 51.2703]	[93.8442, 94.1001, 94.3053]
98	[48.0756, 49.8133, 51.2832]	[94.7219, 94.9422, 95.1194]
128	[48.0569, 49.8143, 51.3021]	[95.3809, 95.5742, 95.7299]
162	[48.0495, 49.8215, 51.3225]	[95.8937, 96.0659, 96.204]
200	[48.0480, 49.8314, 51.3425]	[96.3041, 96.4593, 96.5846]
800	[48.0931, 49.8806, 51.3917]	[98.1518, 98.2297, 98.2928]
1800	[48.0894, 49.8898, 51.4011]	[98.7678, 98.8198, 98.8620]
5000	[48.0942, 49.8946, 51.4040]	[99.2607, 99.2919, 99.3172]

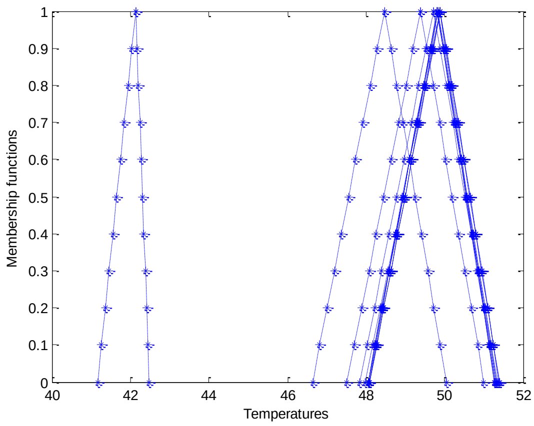

Figure 5.31. Minimum nodal TFN temperatures

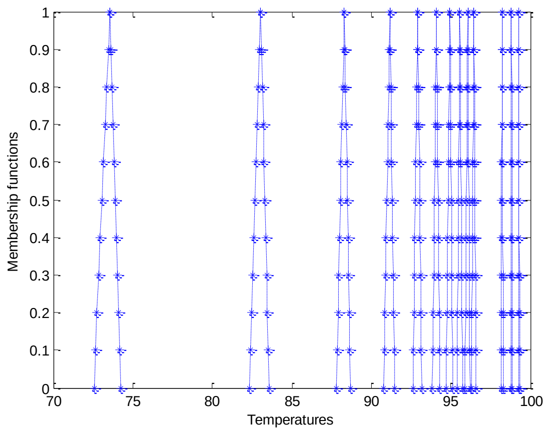

Figure 5.32. Maximum nodal TFN temperatures

5.3 Conclusion

In this chapter, conduction-convection problems have been investigated where the uncertainties are considered as interval/fuzzy. The problems are solved by the proposed fuzzy finite element method. Here two types of fuzzy numbers viz. TFN and TRFN have been considered. The possibility of temperature distribution at any nodal points of the domain may be estimated using the proposed fuzzy finite element method.

Chapter 6

One Group Neutron Diffusion Problems

The contents of this chapter have been published/accepted in the following journals/conferences:

1. S. Chakraverty and S. Nayak, Non probabilistic solution of uncertain neutron diffusion equation for imprecisely defined homogeneous bare reactor, ***Annals of Nuclear Energy***, 2013, Vol. 62, 251–259;
2. S. Nayak and S. Chakraverty, Numerical solution of uncertain neutron diffusion equation for imprecisely defined homogeneous triangular bare reactor, ***Sadhana***, 2015, Accepted;
3. S. Nayak and S. Chakraverty, Fuzzy finite element approach to solve uncertain neutron diffusion equation for imprecisely defined homogeneous rectangular bare reactor, Third International conference of Gwalior Academy of Mathematical Sciences (GAMS) and Third IFIP International conference on Bioinformatics, MANIT Bhopal, 22-26 September, 2013;
4. S. Nayak and S. Chakraverty, Fuzzy Finite Element Method to Investigate Uncertain Neutron Diffusion Problems, Theme Meeting on Fuzzy and Interval Based Uncertainty Modeling, NIT Rourkela, Odisha, India,18-20 July, 2013.

Chapter 6

One Group Neutron Diffusion Problems

The neutron collision inside a reactor depends upon geometry of the reactor, diffusion coefficient and absorption coefficient etc. In general these parameters are not crisp and hence we get uncertain neutron diffusion equation. Here we have investigated the governing one group neutron diffusion equation for bare square and triangular homogeneous reactors. The uncertain governing differential equation has been modelled with imprecisely defined parameters and solved by fuzzy finite element method (Chapter 3). Using fuzzy finite element method, obtained eigenvalues and effective multiplication factors are studied. Corresponding results are compared with the classical finite element method in special cases and various uncertain results have been discussed.

6.1 Imprecisely Defined Homogeneous Bare Square Reactor

In this investigation one group of neutron diffusion equation has been studied. The corresponding eigenvalue for one group of neutron diffusion equation for bare square homogeneous reactor are obtained. The presence of uncertain parameters makes the system uncertain and the uncertain eigenvalues along with the effective multiplication factors are studied in detail. Here we have also considered two types of fuzzy numbers viz. TFN and TRFN to solve the discussed problem.

6.1.1 Formulation of the Problem

As it is known that the principle of neutron conservation can be expressed in a simple form for a system of mono energetic neutrons, multi group equations can be analysed by considering the series of one group equations.

The standard functional for corresponding one group diffusion equation may be written as

$$I(\phi)=\frac{1}{2}\iint_R \left[D\left(\frac{\partial \phi}{\partial x}\right)^2 + D\left(\frac{\partial \phi}{\partial y}\right)^2 + \sigma\phi^2 - 2S\phi \right] dxdy \tag{6.1}$$

where ϕ is a constant over a partial/total portion of the periphery, D is the diffusion coefficient, σ is the absorption coefficient and S is the source term.

If we apply traditional FEM to handle the problem the corresponding domain of the problem is divided into number of subdomain and each of them is called element. For each element we may find the functional and similarly for the entire domain the functional may be found out by summing each functional element wise. The above procedure may be written in the following way.

First the domain R may be represented as

$$R = \sum_{e=1}^{n} R^e \tag{6.2}$$

and the functional $I(\phi)$ is defined as

$$I(\phi) = \sum_{e=1}^{n} I^e(\phi) \tag{6.3}$$

where n is the total number of elements and $I^e(\phi)$ denotes the contribution of element e to the functional $I(\phi)$. Now the Eq. (6.1) for each elemental functional may be written as

$$I^e(\phi) = \frac{1}{2} \iint_R \left[D^e \left(\frac{\partial \phi^e}{\partial x} \right)^2 + D^e \left(\frac{\partial \phi^e}{\partial y} \right)^2 + \sigma^e \phi^2 - 2S^e \phi^e \right] dxdy \tag{6.4}$$

For each element e the scalar flux ϕ^e is approximated by a piecewise interpolation polynomial. Depending on the interpolation polynomial, stiffness matrices are obtained by minimizing the elemental functional $I^e(\phi)$. The stiffness matrices are assembled and finally we get the algebraic form which is represented as

$$[K]\{\phi\} = \{Q\} \tag{6.5}$$

where $[K]$ is the assembled stiffness matrix corresponding to leakage and absorption terms and $\{Q\}$ is the assembled force vector for the source term.

In general when neutrons undergo scattering, the neutron transport equation involves uncertainty. The uncertainty occurs due to the imprecise value of operating parameters viz. geometry, diffusion and absorption coefficients etc. Here these uncertain parameters are taken as fuzzy. To investigate the uncertain spectrum of neutron flux distribution we formulate fuzzy finite element method with linear triangular fuzzy element discretising the domain.

Let us consider that the coordinates of linear triangular elements are in fuzzy and hence we may write

$$\begin{aligned} \tilde{x} &= L_1 \tilde{x}_1 + L_2 \tilde{x}_2 + L_3 \tilde{x}_3; \\ \tilde{y} &= L_1 \tilde{y}_1 + L_2 \tilde{y}_2 + L_3 \tilde{y}_3; \\ \tilde{L} &= \tilde{L}_1 + \tilde{L}_2 + \tilde{L}_3; \end{aligned} \tag{6.6}$$

where $\tilde{L}_i\ (i=1,2,3)$ are nondimensionalized coordinates.

The above Eq. (6.6) in matrix form is represented in the following way

$$\begin{bmatrix} 1 & 1 & 1 \\ \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \\ \tilde{y}_1 & \tilde{y}_2 & \tilde{y}_3 \end{bmatrix} \begin{Bmatrix} \tilde{L}_1 \\ \tilde{L}_2 \\ \tilde{L}_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ \tilde{x} \\ \tilde{y} \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \tilde{L}_1 \\ \tilde{L}_2 \\ \tilde{L}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \\ \tilde{y}_1 & \tilde{y}_2 & \tilde{y}_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ \tilde{x} \\ \tilde{y} \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \tilde{L}_1 \\ \tilde{L}_2 \\ \tilde{L}_3 \end{Bmatrix} = \begin{bmatrix} \tilde{x}_2\tilde{y}_3 - \tilde{x}_3\tilde{y}_2 & \tilde{y}_2 - \tilde{y}_3 & \tilde{x}_3 - \tilde{x}_2 \\ \tilde{x}_3\tilde{y}_1 - \tilde{x}_1\tilde{y}_3 & \tilde{y}_3 - \tilde{y}_1 & \tilde{x}_1 - \tilde{x}_3 \\ \tilde{x}_1\tilde{y}_2 - \tilde{x}_2\tilde{y}_1 & \tilde{y}_1 - \tilde{y}_2 & \tilde{x}_2 - \tilde{x}_1 \end{bmatrix} \begin{Bmatrix} 1 \\ \tilde{x} \\ \tilde{y} \end{Bmatrix}$$

where area of the fuzzy triangle is $\tilde{\Delta} = \dfrac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \\ \tilde{y}_1 & \tilde{y}_2 & \tilde{y}_3 \end{bmatrix}$.

We now denote

$$\tilde{a}_1 = \tilde{x}_3 - \tilde{x}_2,\ \tilde{a}_2 = \tilde{x}_1 - \tilde{x}_3,\ \tilde{a}_3 = \tilde{x}_2 - \tilde{x}_1;$$
$$\tilde{b}_1 = \tilde{y}_2 - \tilde{y}_3,\ \tilde{b}_2 = \tilde{y}_3 - \tilde{y}_1,\ \tilde{b}_3 = \tilde{y}_1 - \tilde{y}_2;$$
$$\tilde{c}_1 = \tilde{x}_2\tilde{y}_3 - \tilde{x}_3\tilde{y}_2,\ \tilde{c}_2 = \tilde{x}_3\tilde{y}_1 - \tilde{x}_1\tilde{y}_3,\ \tilde{c}_3 = \tilde{x}_1\tilde{y}_2 - \tilde{x}_2\tilde{y}_1.$$

If $\tilde{\phi}$ is the flux distribution then it may be written as

$$\tilde{\phi} = \tilde{L}_1\tilde{\phi}_1 + \tilde{L}_2\tilde{\phi}_2 + \tilde{L}_3\tilde{\phi}_3 \tag{6.7}$$

The differentiation and integration formulae are then given by,

$$\frac{\partial}{\partial\tilde{x}} = \sum_{i=1}^{3}\frac{\tilde{b}_i}{2\tilde{\Delta}}\frac{\partial}{\partial\tilde{L}_i}, \quad \frac{\partial}{\partial\tilde{y}} = \sum_{i=1}^{3}\frac{\tilde{a}_i}{2\tilde{\Delta}}\frac{\partial}{\partial\tilde{L}_i} \text{ and } \iint_R \tilde{L}_1^p\tilde{L}_2^q\tilde{L}_3^r\, d\tilde{\Delta} = \frac{p!q!r!}{(p+q+r+2)!}(2\tilde{\Delta}).$$

Hence

$$\frac{\partial\tilde{\phi}^{(e)}}{\partial\tilde{x}} = \frac{1}{2\tilde{\Delta}}\left\{\tilde{b}_1\tilde{\phi}_1 + \tilde{b}_2\tilde{\phi}_2 + \tilde{b}_3\tilde{\phi}_3\right\} = \begin{bmatrix} \dfrac{\tilde{b}_1}{2\tilde{\Delta}} & \dfrac{\tilde{b}_2}{2\tilde{\Delta}} & \dfrac{\tilde{b}_3}{2\tilde{\Delta}} \end{bmatrix}\begin{Bmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \\ \tilde{\phi}_3 \end{Bmatrix}$$

Similarly,

$$\frac{\partial\tilde{\phi}^{(e)}}{\partial\tilde{y}} = \frac{1}{2\tilde{\Delta}}\left\{\tilde{a}_1\tilde{\phi}_1 + \tilde{a}_2\tilde{\phi}_2 + \tilde{a}_3\tilde{\phi}_3\right\} = \begin{bmatrix} \dfrac{\tilde{a}_1}{2\tilde{\Delta}} & \dfrac{\tilde{a}_2}{2\tilde{\Delta}} & \dfrac{\tilde{a}_3}{2\tilde{\Delta}} \end{bmatrix}\begin{Bmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \\ \tilde{\phi}_3 \end{Bmatrix}.$$

Using above formulation one may get the leakage and absorption stiffness matrices. Accordingly, corresponding stiffness matrices of each element for leakage and absorption term is given by,

$$\left[\tilde{K}_1\right] = \frac{\tilde{D}^{(e)}}{4\tilde{\Delta}}\begin{bmatrix} \tilde{a}_1^{\,2}+\tilde{b}_1^{\,2} & \tilde{a}_1\tilde{a}_2+\tilde{b}_1\tilde{b}_2 & \tilde{a}_1\tilde{a}_3+\tilde{b}_1\tilde{b}_3 \\ \tilde{a}_1\tilde{a}_2+\tilde{b}_1\tilde{b}_2 & \tilde{a}_2^{\,2}+\tilde{b}_2^{\,2} & \tilde{a}_2\tilde{a}_3+\tilde{b}_2\tilde{b}_3 \\ \tilde{a}_1\tilde{a}_3+\tilde{b}_1\tilde{b}_3 & \tilde{a}_2\tilde{a}_3+\tilde{b}_2\tilde{b}_3 & \tilde{a}_3^{\,2}+\tilde{b}_3^{\,2} \end{bmatrix} \text{ and } \left[\tilde{K}_2\right] = \frac{\tilde{\sigma}^{(e)}\tilde{\Delta}}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \text{ respectively.}$$

The source vector $\{\tilde{f}\}$ for each element may be written as

$$\{\tilde{f}\} = \frac{\tilde{S}^{(e)}\tilde{\Delta}}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}.$$

6.1.2 Numerical Example

The governing differential equation for the bare homogeneous reactor (Glasstone and Sesonke (2004)) is as follows

$$D\nabla^2\phi + S = \Sigma_a\phi \tag{6.8}$$

The boundary condition is $\phi(x,\pm1.5) = 0 = \phi(\pm1.5, y)$ and it is solved first by classical (traditional) finite element method for the sake of completeness and then fuzzy finite element method is presented. Here the square homogeneous region has been divided into 18 and 72 elements as shown in Figures 6.1, 6.2 and Figures 6.3, 6.4. Two different types of discretizations are considered and the results are compared.

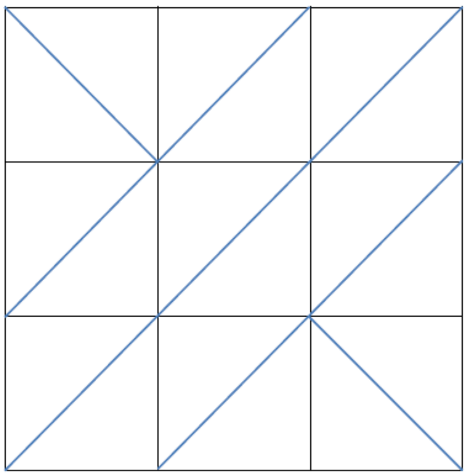

Figure 6.1. 18 elements

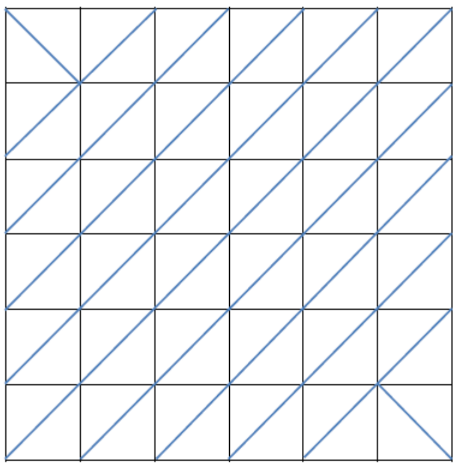

Figure 6.2. 72 elements

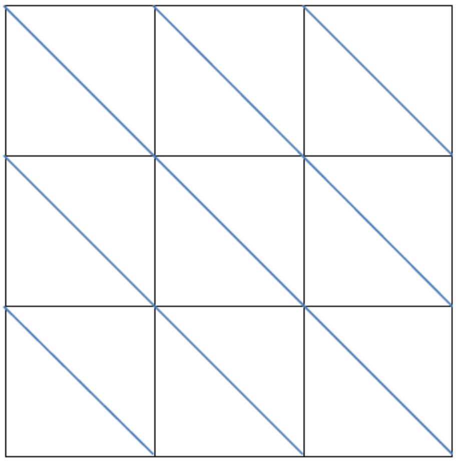

Figure 6.3. 18 elements

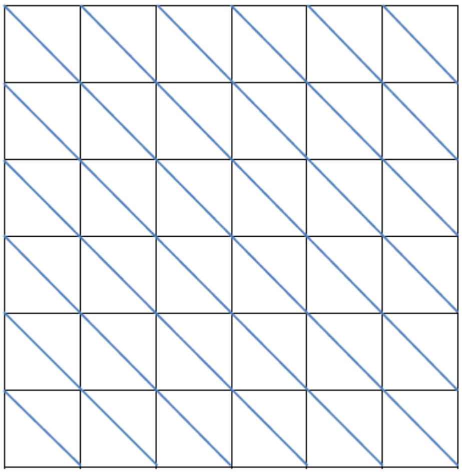

Figure 6.4. 72 elements

Initially the eigenvalues and corresponding effective multiplication factors are investigated when the involved parameters viz. diffusion coefficient (D) and absorption coefficient (σ) are crisp. The different values of above parameters are given in Table 6.1. The observed results for two different types of discretizations are presented in Tables 6.2 to 6.7.

As expected, if there involved some uncertainty then it affects the system. When neutrons undergo diffusion, neutrons suffer scattering collisions with the nuclei assumed to be initially stationary and as a result, a typical neutron trajectory consists of a number of short path elements. These are scattering free paths. The average of these is the mean free path. When a large number of neutrons are considered, there is a net movement of neutrons from regions of higher to those of lower concentration. As it is known that the path of the neutron after a scattering collision may not be given exactly we may take the cross section and transport mean free path as fuzzy. As a result, the diffusion coefficient will lie in an uncertain region and becomes fuzzy. Similar way the absorption coefficient may also be taken as fuzzy. Here we have taken two different types of fuzzy numbers (TFN and TRFN) to handle these uncertainties. The uncertain values along with the crisp are given in Table 6.1.

Table 6.1. Crisp and fuzzy values of the involved parameters

Parameters	Crisp value	TFN	TRFN
Diffusion coefficient	1	$[0.5+0.5\alpha, 1.5-0.5\alpha]$	$[0.5+0.3\alpha, 1.5-0.3\alpha]$
Absorption coefficient	1	$[0.5+0.5\alpha, 1.5-0.5\alpha]$	$[0.5+0.3\alpha, 1.5-0.3\alpha]$

The above uncertain values are used in the developed fuzzy finite element method and eigenvalues are obtained. The uncertain fuzzy eigenvalues under different considerations are shown in Tables 6.2 to 6.7.

Table 6.2. Comparison of eigenvalues when $\sigma = 1$ and $D = [0.5 + 0.5\alpha, 1.5 - 0.5\alpha]$

No. of elements	Classical FEM (Figure and Figure)	Proposed Fuzzy FEM (Figure and Figure)	Classical FEM (Figure and Figure)	Proposed Fuzzy FEM (Figure and Figure)
18	2.6667	$[1.3333 + 1.3334\alpha,$ $4 - 1.3333\alpha]$	2.8195	$[1.4097 + 1.4098\alpha,$ $4.2293 - 1.4098\alpha]$
72	2.3426	$[1.1713 + 1.1713\alpha,$ $3.5140 - 1.1714\alpha]$	2.3454	$[1.1727 + 1.1727\alpha,$ $3.5181 - 1.1727\alpha]$

Table 6.3. Comparison of eigenvalues when $D = 1$ and $\sigma = [0.5 + 0.5\alpha, 1.5 - 0.5\alpha]$

Number of elements	Classical FEM (Figure and Figure)	Proposed Fuzzy FEM (Figure and Figure)	Classical FEM (Figure and Figure)	Proposed Fuzzy FEM (Figure and Figure)
18	2.6667	$[1.7778 + 0.48887\alpha,$ $5.3333 - 2.6666\alpha]$	2.8195	$[1.8797 + 0.9398\alpha,$ $5.6391 - 2.8196\alpha]$
72	2.3426	$[1.5618 + 0.7808\alpha,$ $4.6853 - 2.3427\alpha]$	2.3454	$[1.5636 + 0.7818\alpha,$ $4.6908 - 2.3454\alpha]$

Table 6.4. Comparison of eigenvalues when $D = [0.5 + 0.5\alpha, 1.5 - 0.5\alpha]$ and $\sigma = [0.5 + 0.5\alpha, 1.5 - 0.5\alpha]$

Number of elements	Classical FEM (Figure and Figure)	Proposed Fuzzy FEM (Figure and Figure)	Classical FEM (Figure and Figure)	Proposed Fuzzy FEM (Figure and Figure)
18	2.6667	$[0.889 + 1.7777\alpha,$ $8 - 5.3333\alpha]$	2.8195	$[0.9398 + 1.8797\alpha,$ $8.4587 - 5.6392\alpha]$
72	2.3426	$[0.7809 + 1.5617\alpha,$ $7.0279 - 4.6853\alpha]$	2.3454	$[0.7818 + 1.5636\alpha,$ $7.0363 - 4.6909\alpha]$

Table 6.5. Comparison of eigenvalues when $\sigma = 1$ and $D = [0.5 + 0.3\alpha, 1.5 - 0.3\alpha]$

Number of elements	Classical FEM (Figure and Figure)	Proposed Fuzzy FEM (Figure and Figure)	Classical FEM (Figure and Figure)	Proposed Fuzzy FEM (Figure and Figure)
18	2.6667	$[1.3333 + 0.8\alpha, 4 - 0.8\alpha]$	2.8195	$[1.4097 + 0.8459\alpha, 4.2293 - 0.8458\alpha]$
72	2.3426	$[1.1713 + 0.7028\alpha, 3.5140 - 0.7028\alpha]$	2.3454	$[1.1727 + 0.7036\alpha, 3.5181 - 0.7036\alpha]$

Table 6.6. Comparison of eigenvalues when $D = 1$ and $\sigma = [0.5 + 0.3\alpha, 1.5 - 0.3\alpha]$

Number of elements	Classical FEM (Figure and Figure)	Proposed Fuzzy FEM (Figure and Figure)	Classical FEM (Figure and Figure)	Proposed Fuzzy FEM (Figure and Figure)
18	2.6667	$[1.7778 + 0.4444\alpha, 5.3333 - 2\alpha]$	2.8195	$[1.8797 + 0.4699\alpha, 5.6391 - 2.1147\alpha]$
72	2.3426	$[1.5618 + 0.3904\alpha, 4.6853 - 1.757\alpha]$	2.3454	$[1.5636 + 0.3909\alpha, 4.6908 - 1.759\alpha]$

Table 6.7. Comparison of eigenvalues when $D = [0.5 + 0.3\alpha, 1.5 - 0.3\alpha]$ and $\sigma = [0.5 + 0.3\alpha, 1.5 - 0.3\alpha]$

Number of elements	Classical FEM (Figure and Figure)	Proposed Fuzzy FEM (Figure and Figure)	Classical FEM (Figure and Figure)	Proposed Fuzzy FEM (Figure and Figure)
18	2.6667	$[0.889 + 0.8888\alpha, 8 - 4\alpha]$	2.8195	$[0.9398 + 0.9399\alpha, 8.4587 - 4.2293\alpha]$
72	2.3426	$[0.7809 + 0.7809\alpha, 7.0279 - 3.5139\alpha]$	2.3454	$[0.7818 + 0.7818\alpha, 7.0363 - 3.5181\alpha]$

In view of these tabulated eigenvalues corresponding effective multiplication factors (k_{eff}) are plotted. These are given pictorially in Figures 6.5 to 6.12.

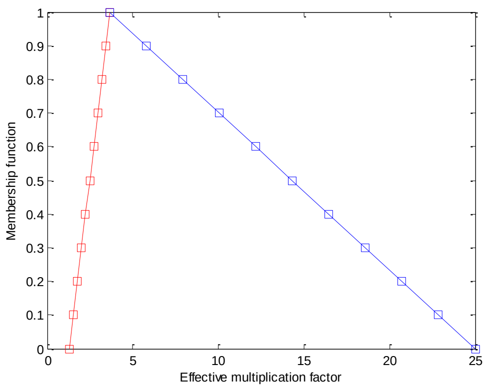

Figure 6.5. TFN for Figure 6.1

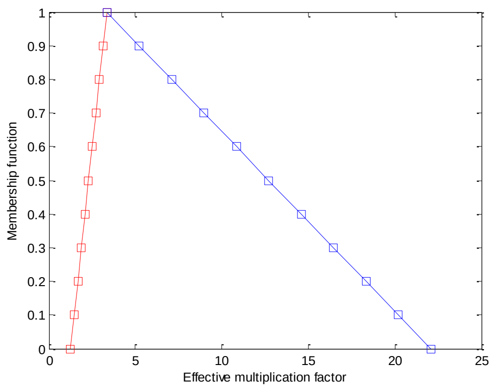

Figure 6.6. TFN for Figure 6.2

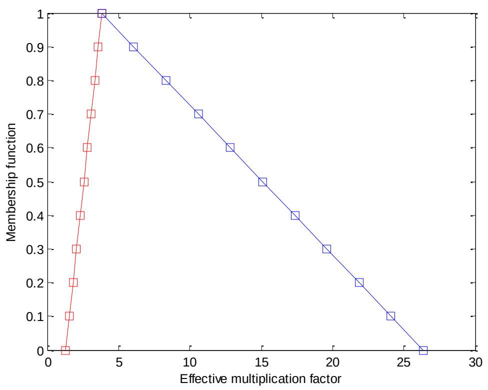

Figure 6.7. TFN for Figure 6.3

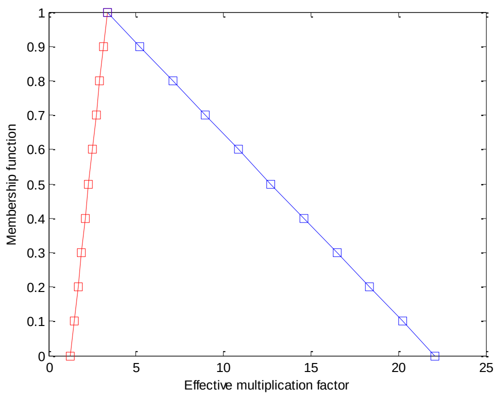

Figure 6.8. TFN for Figure 6.4

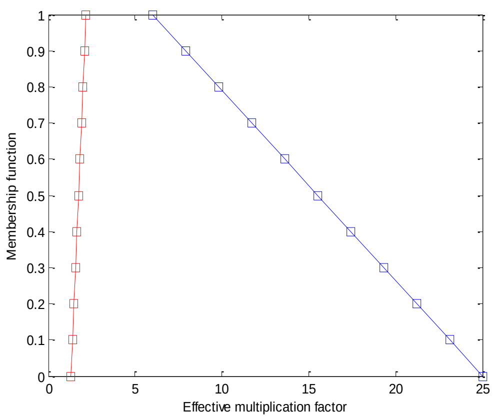

Figure 6.9. TRFN for Figure 6.1

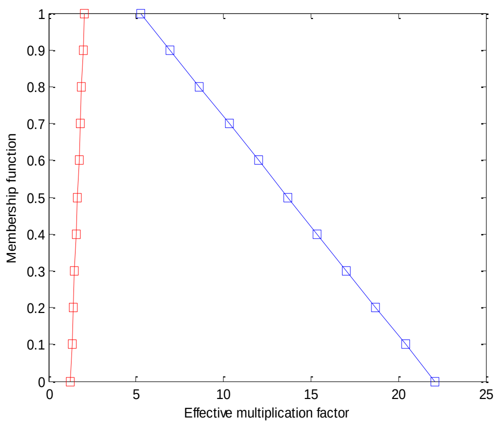

Figure 6.10. TRFN for Figure 6.2

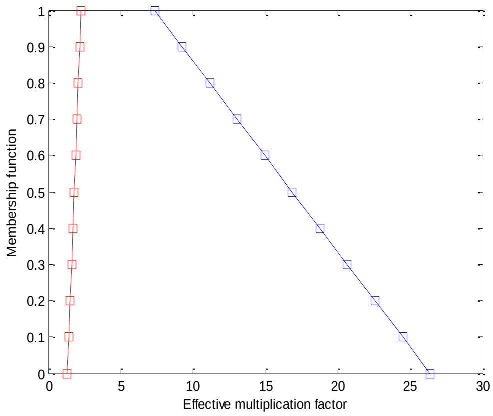

Figure 6.11. TRFN for Figure 6.3

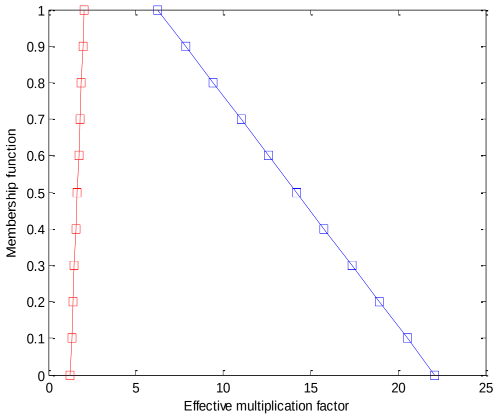

Figure 6.12. TRFN for Figure 6.4

6.1.3 Discussion

When neutrons undergo diffusion in reactor then it suffers scattering collisions with the nuclei assumed to be initially stationary and they make different trajectory paths. Hence, to study the eigenvalue problem for corresponding one group neutron diffusion equation, we have considered two different types of discretizations of a bare square homogeneous reactor. Initially the eigenvalue problem is solved by classical finite element method for crisp parameters and following important known points may be reported.

- We find that there is a variation in eigenvalues when different discretizations with same number of elements are taken. Here it is observed that a better approximation to eigenvalues is found for Figures 6.1 and 6.2.
- In Figures 6.1 and 6.2 there are two distinct nodes where the contribution values towards the stiffness matrix is more as compare to Figures 6.3 and 6.4.
- It is observed that if the contribution value towards the stiffness matrix increases with same number of elements then better approximation for eigenvalues occurs. The geometry in Figures 6.1 and 6.2 gives better results.
- It is found that in both the cases the eigenvalues converging with respect to the increasing in number of elements.
- In reference to respective eigenvalues, corresponding effective multiplication factors also converges.

In general there involves uncertainty in system and these uncertainty occurs due to the involve parameters viz. diffusion and absorption coefficients. To handle such uncertainty an alternate fuzzy finite element method has been used here. Two different types of fuzzy numbers such as TFN and TRFN have been considered to investigate the uncertain eigenvalues and effective multiplication factors. The detail investigation is summarised as indicated below.

- We get different spectrum of eigenvalues for different discretizations (Figures 6.1, 6.2 and Figures 6.3, 6.4).
- When the parameters are taken as fuzzy, it is observed that the absorption coefficient is more sensitive.
- From Tables 6.2, 6.3, 6.5 and 6.6 it is found that when only absorption coefficient is taken as fuzzy then the uncertain bound for eigenvalues is wide in comparison with the case where only diffusion coefficient is taken as fuzzy.
- When both the parameters are taken as fuzzy then there is much larger bound of uncertain eigenvalues than the previous cases. These results are depicted in Tables 6.4 and 6.7.
- The effective multiplication factors for corresponding eigenvalues of Tables 6.4 and 6.7 are plotted in Figures 6.5 to 6.12.
- In view of the above Figures 6.5 to 6.12, we conclude that the uncertain bound of effective multiplication factors increase drastically.
- It is also seen that the geometry of the discretization plays a significant role. So we may choose a better geometry to get better distribution of uncertain effective multiplication factors.
- As mentioned above depending upon the value of resulting uncertain effective multiplication factors the neutron density fluctuates.

It may be a point to be noted that the reliability of the fuzzy results can be seen in the special cases viz. crisp and interval which are derived from the fuzzy values. As such three cases are reported with respect to the above.

Case-1

Here we have considered only left monotonic increasing functions of the resultant effective multiplication factors. The resulting effective multiplication factors vary with the value of membership functions. Assigning zero for the value of α we get the left bound of the uncertain fuzzy effective multiplication factors. Similarly if the value of α is taken as one then we get right bound of the left monotonic increasing functions which are the centre value of TFN.

Case-2

In this case only right monotonic decreasing functions of the resultant effective multiplication factors are considered. Resulting effective multiplication factors vary with the value of membership functions. Assigning zero for the value of α we get the right bound of the uncertain fuzzy effective multiplication factors. Similarly if the value of α is taken as one then we get left bound of the right monotonic decreasing functions which are the centre value of TFN.

Case-3

In this case let us consider the value of α is one for both the left and right monotonic functions. We observe that the resultant effective multiplication factors become same for both monotonic functions. If we consider TFN then we find that it is nothing but the centre value. For TRFN we get two different values for both monotonic functions. Here we get an interval of effective multiplication factors where the membership functions are normalised.

6.2 Imprecisely Defined Homogeneous Triangular Bare Reactor

Here we have considered one group of neutron diffusion equation for triangular bare reactor. The corresponding eigenvalues for one group of neutron diffusion equation for bare triangular homogeneous reactor is investigated. The presence of uncertain parameters makes the system uncertain and the uncertain eigenvalues are studied. To handle uncertain system, the problem is modelled and proposed fuzzy finite element method has been used.

6.2.1 Numerical Example

The governing differential equation for the bare homogeneous reactor (Glasstone & Sesonke 2004) is as follows

$$D\nabla^2\phi + S = \Sigma_a\phi \tag{6.9}$$

We have considered a triangular (equilateral) bare reactor having each side of 4 units and it is discretized into triangular element as given in Figure 6.13.

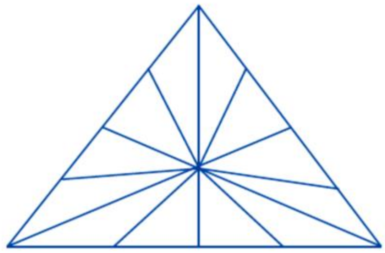

Figure 6.13. Triangular element discretization of triangular plate

Here the parameters such as diffusion and absorption coefficients are considered as fuzzy which are presented in Table 6.8.

Table 6.8. Triangular fuzzy numbers for uncertain parameters

Parameters	Crisp value	TFN
Diffusion coefficient	1	$[0.5+0.5\alpha, 1.5-0.5\alpha]$
Absorption coefficient	1	$[0.5+0.5\alpha, 1.5-0.5\alpha]$

Initially the governing one group neutron diffusion equation is solved by considering only crisp parameters and then the proposed method for the modelled uncertain one group neutron diffusion equation is solved. Eigenvalues for both the crisp and fuzzy parameters are obtained and the values are depicted in Table 6.9 for different number of element discretization in the FEM and FFEM.

Table 6.9. Crisp and triangular fuzzy eigenvalues for triangular plate

Number of elements	Crisp eigenvalues	Triangular fuzzy eigenvalues
6	0.6425	[0.6377, 0.6425, 0.647]
12	0.6264	[0.6236, 0.6264, 0.6297]
24	0.526	[0.5251, 0.526, 0.527]
48	0.5083	[0.508, 0.5083, 0.5087]
96	0.5034	[0.5032, 0.5034, 0.5036]
192	0.5015	[0.5015, 0.5015, 0.5016]
384	0.5007	[0.5007, 0.5007, 0.5008]
1536	0.5002	[0.5002, 0.5002, 0.5002]

For better visualization of the obtained results, eigenvalues for different number of discretizations of the domain are plotted and shown in Figures 6.14 to 6.21.

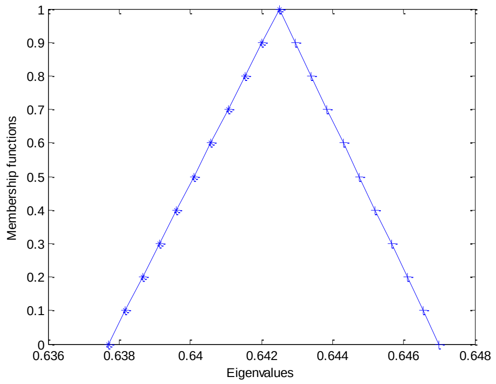

Figure 6.14. 6 elements discretization

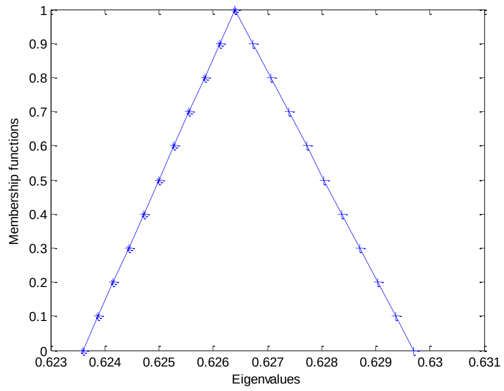

Figure 6.15. 12 elements discretization

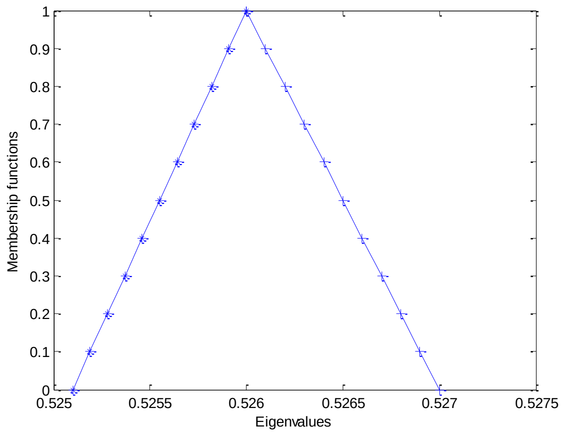

Figure 6.16. 24 elements discretization

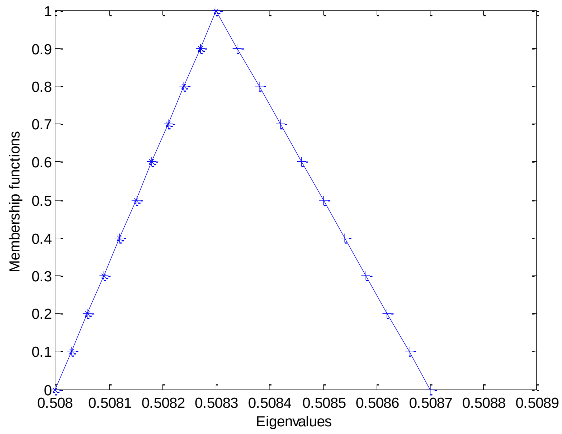

Figure 6.17. 48 elements discretization

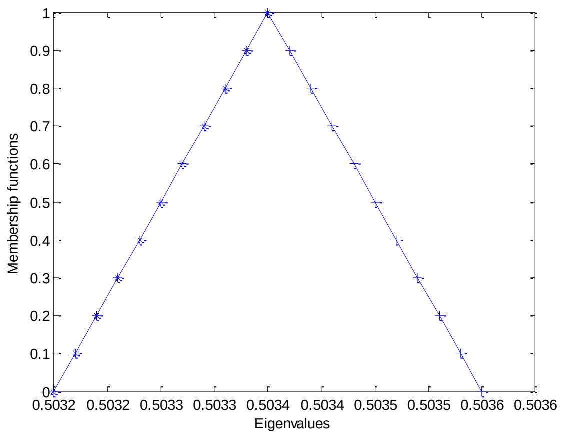

Figure 6.18. 96 elements discretization

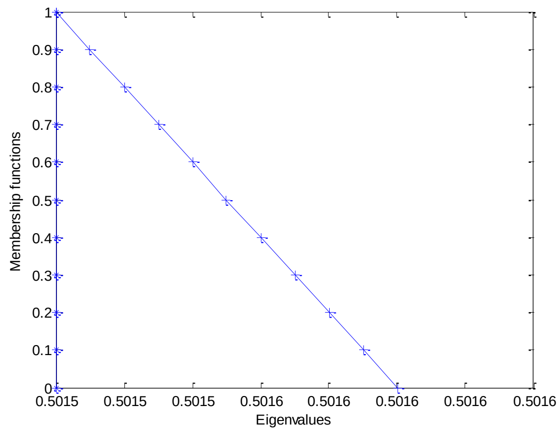

Figure 6.19. 192 elements discretization

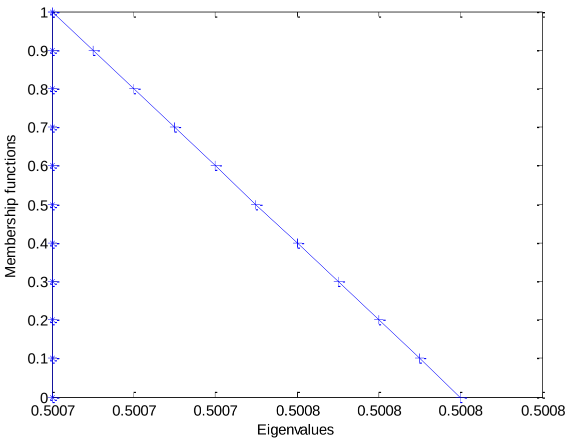

Figure 6.20. 384 elements discretization

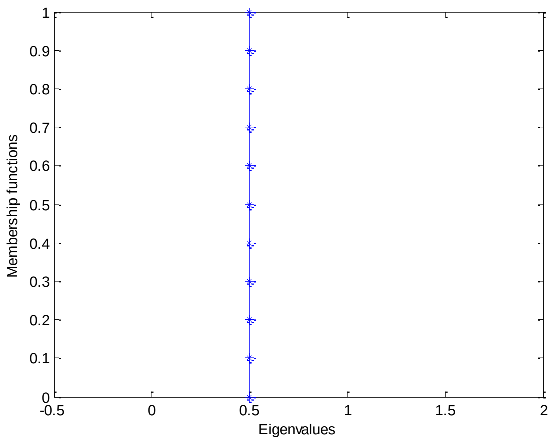

Figure 6.21. 1536 elements discretization

The variation of both the crisp and fuzzy eigenvalues may be studied from Figure 6.22. Here a set of eigenvalues are given and the convergence is studied.

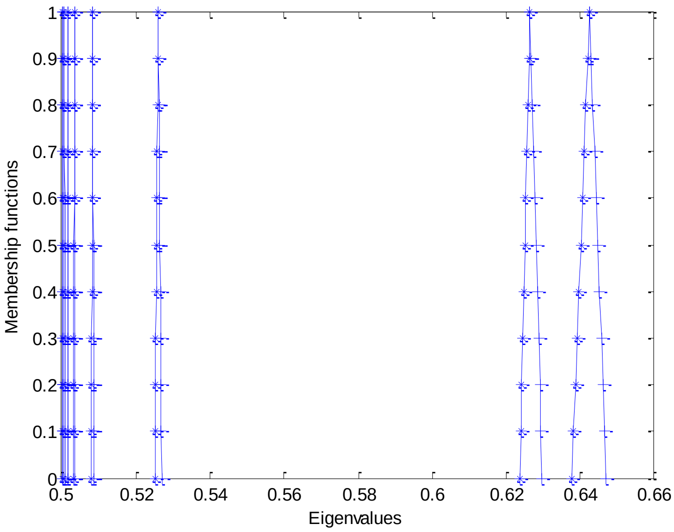

Figure 6.22. Triangular fuzzy numbers for various element discretizations

6.2.2 Discussion

When neutrons undergo diffusion in reactor then it suffers scattering collisions with the nuclei assumed to be initially stationary and they make different trajectory paths. Hence, to study the eigenvalue problem for corresponding one group neutron diffusion equation, we have considered triangular element discretization for one group neutron diffusion equation for a triangular bare reactor. Initially the eigenvalue problem is solved by classical finite element method for crisp parameters and then it is solved by proposed fuzzy finite element procedure.

Here triangular bare reactor is considered and neutron flux at centre of the triangular geometry is taken as zero. The geometry is discretized into number of triangular elements as given in Figure 6.13. So the neutron flux distributions are studied for other nodal points. Solving eigenvalue problem we get a set of eigenvalues for different number of element discretizations. It is seen that the eigenvalues are converging with respect to the increase in number of elements for discretized triangular bare reactor. The pattern of the convergence is presented in Table 6.9.

The diffusion and absorption coefficients are taken as uncertain. These uncertain parameters are considered as TFN to investigate the uncertain eigenvalues. Again the boundary condition for this uncertain case is taken same as that of the crisp case. As we increase the number of elements the uncertain eigenvalues get converged. Further, uncertain width of eigenvalues decreases with increase in number of discretizations of the triangular bare homogeneous reactor.

From the obtained results it is observed that the shape of the TFN changes as we discretise the domain into more number of elements and these are shown in Figures 6.14 to 6.21. This variation of TFN occurs due to the left, right and centre values of the TFN. As we move on with the increase in number of discretization of the domain we get right angular shaped fuzzy number. The trends of the shapes are shown in Figures 6.14 to 6.21. In Figures 6.19 and 6.20 the left and centre values are obtained as same so we get left monotonically increasing function parallel to membership functions axis and width of the left bound from centre becomes zero. In Figure 6.21, the left, right and centre values approximately coincide and hence we get a straight line parallel to membership function axis. Here the variations of eigenvalues become constant with the change of membership functions. From Figure 6.22, it is seen that if we go on increasing the number of discretization of the said domain we get a series of uncertain eigenvalues and these triangular fuzzy eigenvalues converges to a constant value.

It may be noted that the reliability of the fuzzy results can be seen in the special cases viz. crisp and interval which are derived from the fuzzy values. As such three cases are reported with respect to the above.

Case-1

Here we have considered only left monotonic increasing functions of the resultant eigenvalues. The resulting eigenvalues vary with the value of membership functions. Assigning zero for the value of α we get the left bound of the uncertain fuzzy eigenvalues. Similarly if the value of α is taken as one then we get right bound of the left monotonic increasing functions which are the centre value of TFN.

Case-2

In this case only right monotonic decreasing functions of the resultant eigenvalues are considered. Resulting eigenvalues vary with the value of membership functions. Assigning zero for the value of α we get the right bound of the uncertain fuzzy eigenvalues. Similarly if the value of α is taken as one then we get left bound of the right monotonic decreasing functions which are the centre value of TFN.

Case-3

In this case let us consider the value of α is one for both the left and right monotonic functions. We observe that the resultant eigenvalues become same for both monotonic functions. If we consider TFN then we find that it is nothing but the centre value. Here we get an interval of eigenvalues where the membership functions are normalised.

6.3 Conclusion

Here one group neutron diffusion equation is investigated. The presence of uncertain parameters makes the system uncertain and the corresponding eigenvalues along with the effective multiplication factors are systematically studied. Again, the problem is handled by the proposed fuzzy finite element method. The procedure may easily be extended to solve other neutron diffusion problems.

Chapter 7

Multi Group Neutron Diffusion Problem

The contents of this chapter have been published in the following journals:

1. S. Nayak and S. Chakraverty, Fuzzy finite element analysis of multigroup neutron diffusion equation with imprecise parameters, ***International Journal of Nuclear Energy Science and Technology***, 2015, Vol. 9, No. 1, 1-22;
2. S. Nayak and S. Chakraverty, A new approach to solve fuzzy system of linear equations, ***Journal of mathematics and computer science***, 2013, Vol. 7, 205-212.

Chapter 7

Multigroup Neutron Diffusion Problem

Present investigation based on the solution of multigroup neutron diffusion equation under uncertain environment. Here multigroup neutron diffusion equation for steady state case is considered and a two group neutron diffusion equation for an example problem is investigated by using proposed fuzzy finite element analysis. An example benchmark problem is demonstrated with uncertain parameters. Various parameters such as thermal conductivity, diffusion, group fission and neutron interaction constants are taken as fuzzy and the uncertain solutions viz. thermal and fast group fluxes are discussed. Finally, obtained results are compared with existing ones in special cases and sensitivity of uncertain parameters are analysed.

7.1 Fuzzy Finite Element for Coupled Differential Equations

Let us consider the standard fuzzy coupled differential equations in α-cut form as follows

$$\begin{aligned}
\frac{d^2[\underline{\phi_1}(\alpha),\overline{\phi_1}(\alpha)]}{dx^2} &= [\underline{c_{11}}(\alpha),\overline{c_{11}}(\alpha)][\underline{\phi_1}(\alpha),\overline{\phi_1}(\alpha)]+[\underline{c_{12}}(\alpha),\overline{c_{12}}(\alpha)][\underline{\phi_2}(\alpha),\overline{\phi_2}(\alpha)] \\
\frac{d^2[\underline{\phi_2}(\alpha),\overline{\phi_2}(\alpha)]}{dx^2} &= [\underline{c_{21}}(\alpha),\overline{c_{21}}(\alpha)][\underline{\phi_1}(\alpha),\overline{\phi_1}(\alpha)]+[\underline{c_{22}}(\alpha),\overline{c_{22}}(\alpha)][\underline{\phi_2}(\alpha),\overline{\phi_2}(\alpha)]
\end{aligned} \tag{7.1}$$

Eq. (7.1) may be modified and write in a compact form in the following way

$$\begin{aligned}
\frac{d^2\tilde{\phi}_1}{dx^2} &= \tilde{c}_{11}\tilde{\phi}_1+\tilde{c}_{12}\tilde{\phi}_2 \\
\frac{d^2\tilde{\phi}_2}{dx^2} &= \tilde{c}_{21}\tilde{\phi}_1+\tilde{c}_{22}\tilde{\phi}_2
\end{aligned} \tag{7.2}$$

where '~' represents the fuzzy numbers.

$\tilde{c}_{ij}, i.j=1,2$ and $\tilde{\phi}_i, i=1,2$ are the coefficients and flux generated in the system.

To find the approximate uncertain flux (ϕ) we have used Galarkin's weighted residue method. Considering linear element discretization the shape functions would be $\left[\tilde{N}_1 \quad \tilde{N}_2\right]^T=\left[1-\frac{\tilde{x}}{l} \quad \frac{\tilde{x}}{l}\right]^T$, l is the length of each element of the domain. Now multiply the shape functions with the residue of Eq. (7.2) we get

$$\tilde{N}_i\left(\frac{d^2\tilde{\phi}_1}{dx^2}-\tilde{c}_{11}\tilde{\phi}_1-\tilde{c}_{12}\tilde{\phi}_2\right)=0$$
$$\tilde{N}_i\left(\frac{d^2\tilde{\phi}_2}{dx^2}-\tilde{c}_{21}\tilde{\phi}_1-\tilde{c}_{22}\tilde{\phi}_2\right)=0,\ i=1,2 \tag{7.3}$$

Integrating Eq. (7.3) over the domain Ω we get

$$\int_\Omega \tilde{N}_i\left(\frac{d^2\tilde{\phi}_1}{dx^2}-\tilde{c}_{11}\tilde{\phi}_1-\tilde{c}_{12}\tilde{\phi}_2\right)=0$$
$$\int_\Omega \tilde{N}_i\left(\frac{d^2\tilde{\phi}_2}{dx^2}-\tilde{c}_{21}\tilde{\phi}_1-\tilde{c}_{22}\tilde{\phi}_2\right)=0,\ i=1,2 \tag{7.4}$$

Solving Eq. (7.4), set of algebraic equations are obtained which further simplification gives the uncertain neutron flux ($\tilde{\phi}$). The above proposed idea has been extended and used for the formulation of multigroup neutron diffusion equations.

7.2 Fuzzy Multigroup Neutron Diffusion Equation

Let us consider one-dimensional reactor core which is divided into various energy groups and different regions having constant material properties. The discretization of the reactor core into various groups is shown in Figure 7.1.

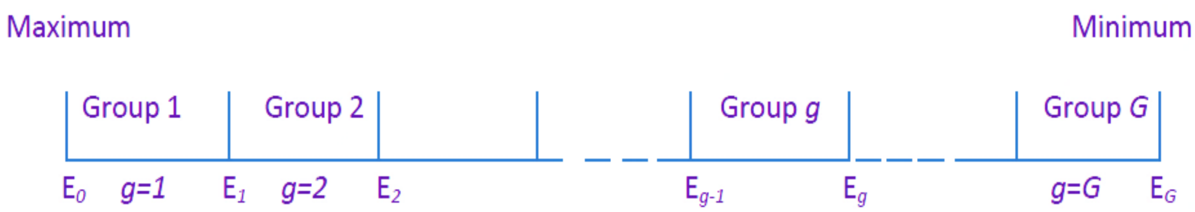

Figure 7.1. Divisions of neutron into G groups

The general form of the fuzzy neutron diffusion equation may be written as (Glasstone & Sesonke 2004)

$$\frac{d}{dx}\left[\tilde{D}_g(x)\frac{d\tilde{\phi}_g(x)}{dx}\right]-\sum_{g}^{t}(x)\tilde{\phi}_g(x)+\sum_{g'=1}^{G}\Sigma^{s}_{g'\to g}(x)\tilde{\phi}_{g'}(x)+\tilde{F}_g(x)=0 \tag{7.5}$$

where, $g=1,2,\ldots G$ and '~' denotes the fuzziness.

In Eq. (7.5), first, second, third and fourth terms represents leakage of neutrons from group g, total rate of neutron interaction, scattering of neutrons from other groups into group g and the rate at which neutrons are produced in the group respectively.

Using above procedure the shape functions are multiplied with Eq. (7.5) and minimized to get

$$[\tilde{N}_g]\frac{d}{dx}\left[\tilde{D}_g(x)\frac{d\tilde{\phi}_g(x)}{dx}\right]-[\tilde{N}_g]\sum_g^t(x)\tilde{\phi}_g(x)+[\tilde{N}_g]\sum_{g'=1}^G\sum_{g'\to g}^s(x)\tilde{\phi}_{g'}(x)+[\tilde{N}_g]\tilde{F}_g(x)=0 \qquad (7.6)$$

Integrating Eq. (7.6) over the domain we get

$$\int_\Omega\left([\tilde{N}_g]\frac{d}{dx}\left[\tilde{D}_g(x)\frac{d\tilde{\phi}_g(x)}{dx}\right]-[\tilde{N}_g]\sum_g^t(x)\tilde{\phi}_g(x)+[\tilde{N}_g]\sum_{g'=1}^G\sum_{g'\to g}^s(x)\tilde{\phi}_{g'}(x)+[\tilde{N}_g]\tilde{F}_g(x)\right)dv=0 \qquad (7.7)$$

Further simplification of Eq. (7.7) gives system of algebraic equations. In matrix form this algebraic equations look like

$$[\tilde{K}_g]\{\tilde{\phi}\}=[\tilde{Q}] \qquad (7.8)$$

Here $[\tilde{K}_g]$ is the fuzzy stiffness matrix for the coupled fuzzy neutron diffusion equation and $\{\tilde{\phi}\}$ is the fuzzy neutron flux vector. In steady case $[\tilde{Q}]$ matrix is zero. Eq. (7.8) is a fully fuzzy system of equations which is tedious to handle. But this difficulty may be overcome by using the method discussed in (Militão et al. 2012).

Now let us consider a two group fuzzy neutron diffusion equation (Wood, De & Oliveira 1984)

$$\begin{aligned}-\frac{d^2\tilde{\phi}_1}{dx^2}&=\tilde{m}_{11}\tilde{\phi}_1+\tilde{m}_{12}\tilde{\phi}_2\\-\frac{d^2\tilde{\phi}_2}{dx^2}&=\tilde{m}_{21}\tilde{\phi}_1+\tilde{m}_{22}\tilde{\phi}_2\end{aligned} \qquad (7.9)$$

where $\tilde{m}_{11}=\dfrac{\left(\tilde{v}\dfrac{\tilde{\Sigma}_{f1}}{k}-\tilde{\Sigma}_{r1}\right)}{\tilde{D}_1}$, $\tilde{m}_{12}=\dfrac{\tilde{v}\tilde{\Sigma}_{f2}}{\tilde{D}_1 k}$, $\tilde{m}_{21}=\dfrac{\tilde{\Sigma}_{12}}{\tilde{D}_2}$, $\tilde{m}_{22}=-\dfrac{\tilde{\Sigma}_{a2}}{\tilde{D}_2}$.

Using the above Galarkin fuzzy finite element formulation for each element having length l we get the following stiffness matrix

$$\begin{bmatrix}\frac{2}{l}+\tilde{m}_{11}\frac{l}{3} & \tilde{m}_{12}\frac{l}{3} & -\frac{2}{l}+\tilde{m}_{11}\left(\frac{l}{2}-\frac{l}{3}\right) & \tilde{m}_{12}\left(\frac{l}{2}-\frac{l}{3}\right)\\ \tilde{m}_{21}\frac{l}{3} & \frac{2}{l}+\tilde{m}_{21}\frac{l}{3} & \tilde{m}_{21}\left(\frac{l}{2}-\frac{l}{3}\right) & -\frac{2}{l}+\tilde{m}_{11}\left(\frac{l}{2}-\frac{l}{3}\right)\\ \tilde{m}_{11}\left(\frac{l}{2}-\frac{l}{3}\right) & \tilde{m}_{12}\left(\frac{l}{2}-\frac{l}{3}\right) & \tilde{m}_{11}\frac{l}{3} & \tilde{m}_{12}\frac{l}{3}\\ \tilde{m}_{21}\left(\frac{l}{2}-\frac{l}{3}\right) & \tilde{m}_{22}\left(\frac{l}{2}-\frac{l}{3}\right) & \tilde{m}_{21}\frac{l}{3} & \tilde{m}_{22}\frac{l}{3}\end{bmatrix}$$

where, $\tilde{m}_{ij}, i,\ j=1, 2$ fuzzy numbers.

In this chapter we have taken an example problem to quantify the uncertain neutron flux for two group neutron diffusion equation.

7.3 Case Study

Let us consider ANL-BSS-6-A2 benchmark problem for steady state case. The core geometry is one dimensional and its length is considered to be 240 cm, which is divided into three regions viz. 40 cm, 160 cm and 40 cm respectively. The boundary flux is zero, which is shown in Figure 7.2. Parameters used in this problem are encrypted in Table 7.1.

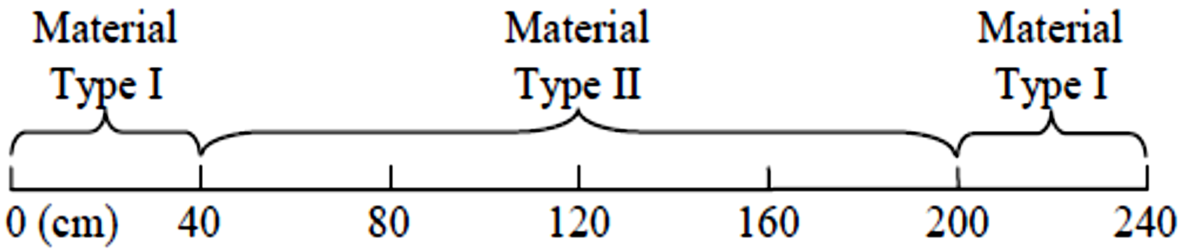

Figure 7.2. Geometry of the ANL-BSS-6-A2 benchmark problem in one dimension

Table 7.1. Crisp and fuzzy parameters for different regions of the domain

Parameters	Region 1,3		Region 2	
	Crisp	Triangular Fuzzy Number (TFN)	Crisp	Triangular Fuzzy Number (TFN)
D_1	1.5	[1, 1.5, 2]	1	[0.5, 1, 1.5]
D_2	0.5	[0.2, 0.5, 0.8]	0.5	[0.2, 0.5, 0.8]
Σ_{r1}	0.26	[0.2, 0.26, 0.32]	0.2	[0.1, 0.2, 0.3]
Σ_{a2}	0.18	[0.15, 0.18, 0.21]	0.08	[0.03, 0.08, 0.13]
Σ_{12}	0.015	[0.01, 0.015, 0.02]	0.01	[0.005, 0.01, 0.15]
$v\Sigma_{f1}$	0.01	[0.005, 0.01, 0.15]	0.005	[0.001, 0.005, 0.009]
$v\Sigma_{f2}$	0.2	[0.15, 0.2, 0.25]	0.099	[0.05, 0.099, 0.148]
k	1	1	1	1

Initially this problem is solved for crisp parameters and then the uncertain variations of parameters are handled. Traditional Galarkin finite element method has been used to investigate the problem with crisp parameters and the obtained neutron fluxes are depicted in Figures 7.3 and 7.4.

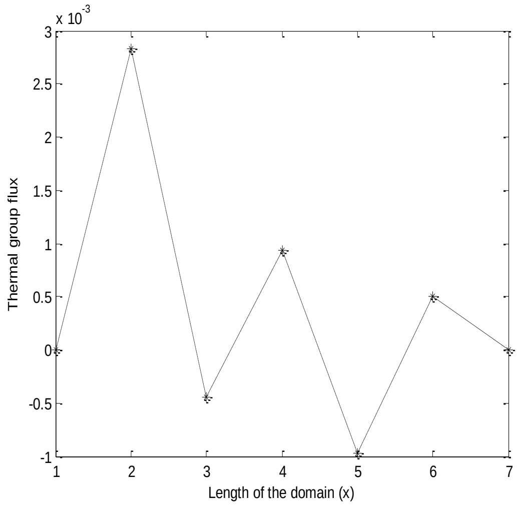

Figure 7.3. Thermal group flux along the domain (with crisp parameters)

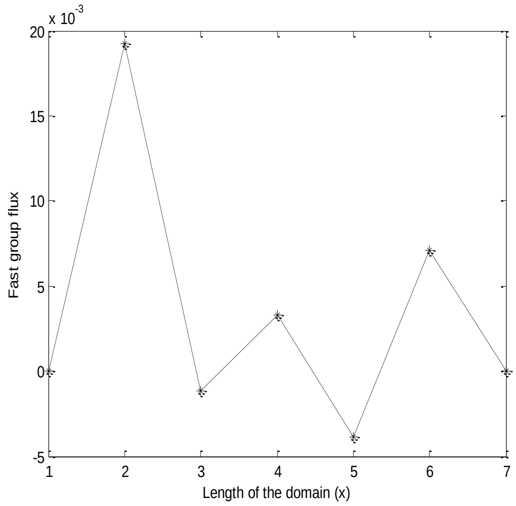

Figure 7.4. Fast group flux along the domain (with crisp parameters)

Obtained uncertain neutron fluxes are shown in Figures 7.5 and 7.6. In these figures, uncertain thermal and fast group fluxes are graphically presented.

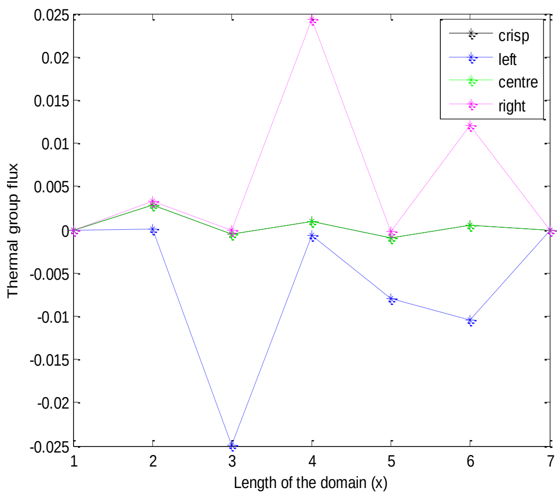

Figure 7.5. Thermal group flux along the domain when all the parameters are fuzzy

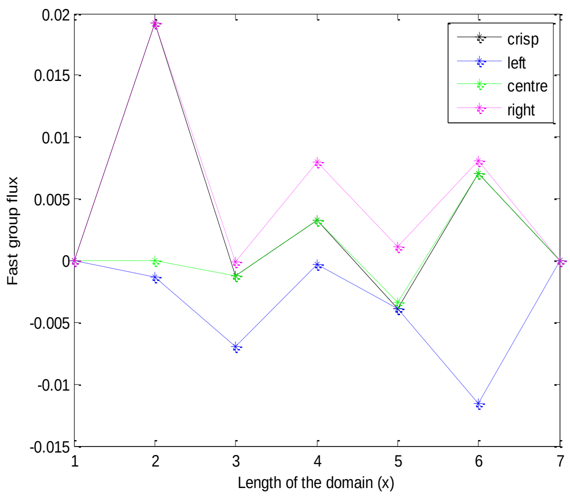

Figure 7.6. Fast group flux along the domain when all the parameters are fuzzy

For different values of α we get different interval values for the uncertain parameters. The variation of flux is graphically encrypted in Figures 7.7 and 7.8 respectively when all the parameters are fuzzy and α is zero.

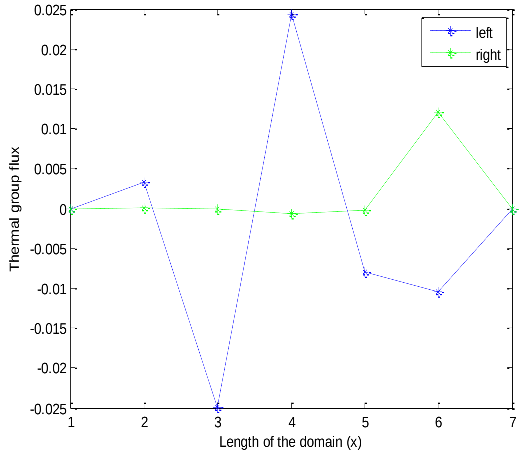

Figure 7.7. Thermal group flux at α is zero when all the parameters are fuzzy

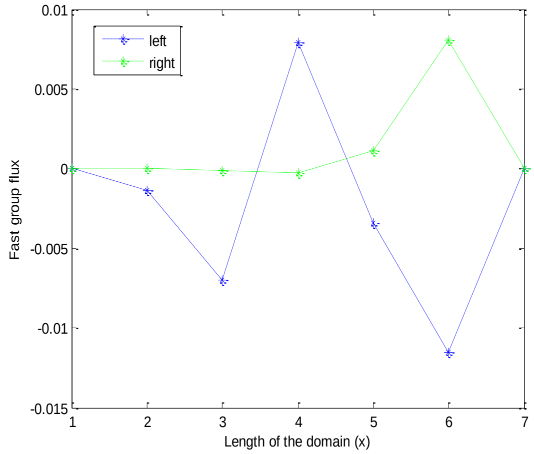

Figure 7.8. Fast group flux at α is zero when all the parameters are fuzzy

In order to see the sensitiveness of the parameters we have considered different cases viz. (i) only diffusion coefficients (D_1, D_2), (ii) neutron interaction coefficients ($\Sigma_{r1}, \Sigma_{a2}, \Sigma_{12}$) and (iii) only group fission constants ($\nu\Sigma_{f1}, \nu\Sigma_{f2}$) are fuzzy. The uncertain neutron fluxes for the above cases are presented in Figures 7.9 to 7.14.

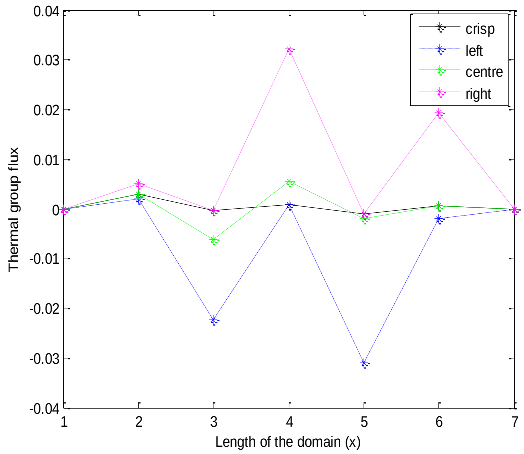

Figure 7.9. Thermal group flux when only D_1, D_2 are fuzzy

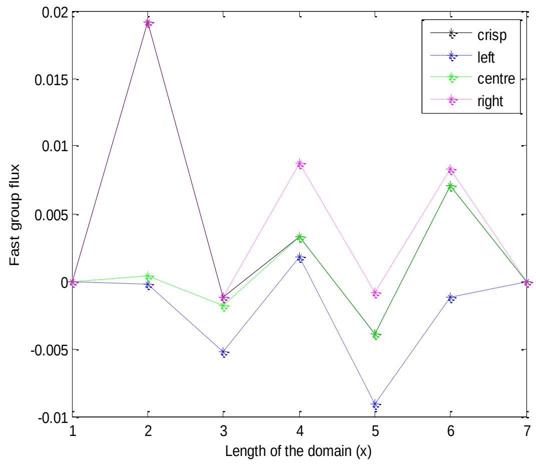

Figure 7.10. Fast group flux when only D_1, D_2 are fuzzy

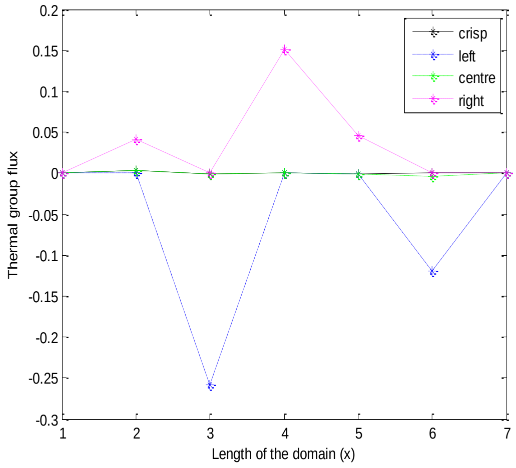

Figure 7.11. Thermal group flux when only $\Sigma_{r1}, \Sigma_{a2}, \Sigma_{12}$ are fuzzy

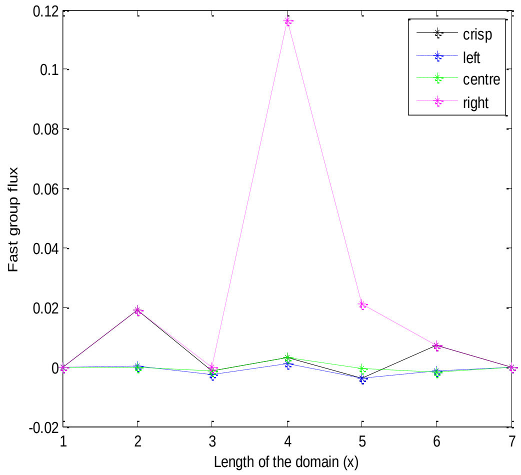

Figure 7.12. Fast group flux when only $\Sigma_{r1}, \Sigma_{a2}, \Sigma_{12}$ are fuzzy

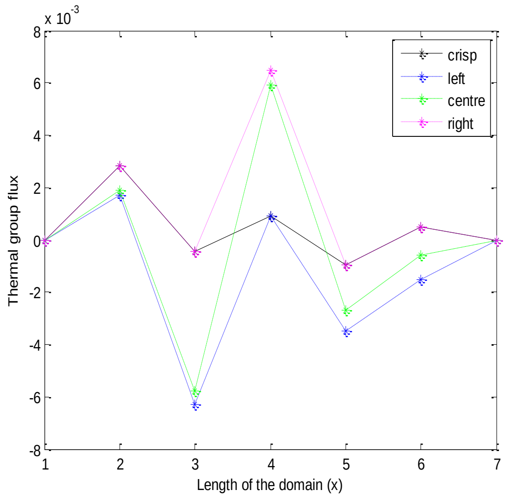

Figure 7.13. Thermal group flux when only $v\Sigma_{f1}, v\Sigma_{f2}$ are fuzzy

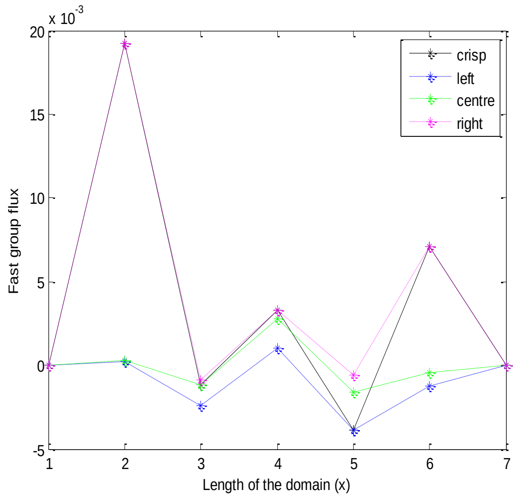

Figure 7.14. Fast group flux when only $v\Sigma_{f1}, v\Sigma_{f2}$ are fuzzy

7.4 Results and Discussion

As mentioned earlier, the above problem is investigated first, for crisp parameters and then uncertainties are considered to demonstrate the proposed fuzzy finite element method. Sensitivity of the uncertain parameters are analysed by considering left, right and centre values for the obtained neutron flux. Some of the major issues are discussed as follows.

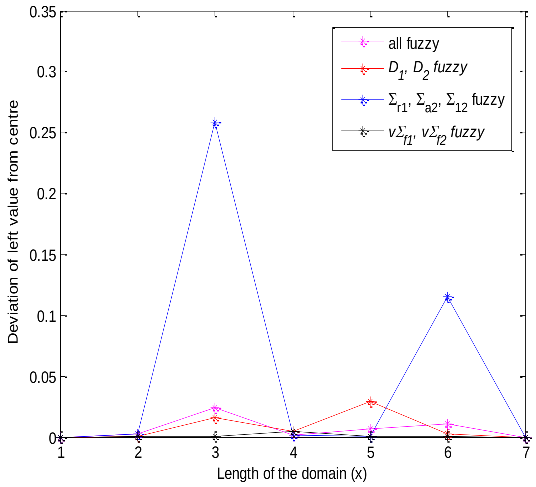

Figure 7.15. Deviation in thermal group flux of left value from centre value of the uncertain neutron fluxes

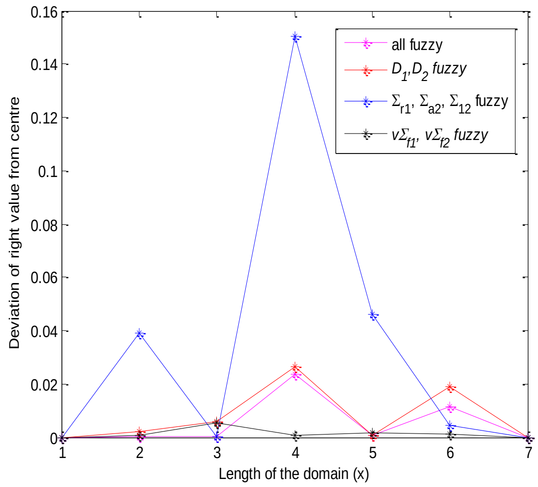

Figure 7.16. Deviation in thermal group flux of right value from centre value of the uncertain neutron fluxes

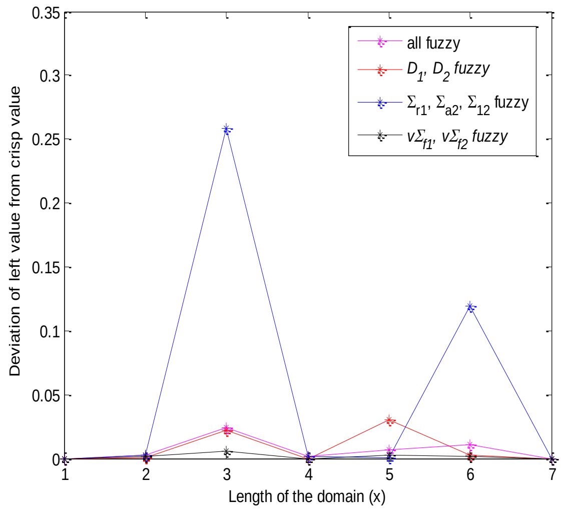

Figure 7.17. Deviation in thermal group flux of left value from crisp value

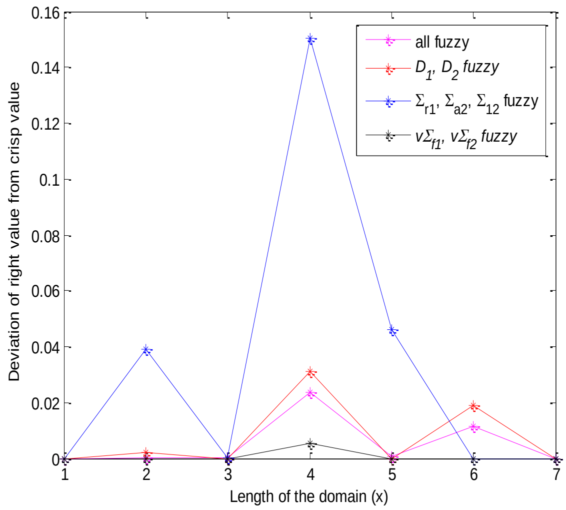

Figure 7.18. Deviation in thermal group flux of right value from crisp value

i) Thermal group neutron fluxes

When only the diffusion coefficients (D_1, D_2) are fuzzy and other parameters are crisp, then in the region 0 cm to 40 cm left, centre and crisp values are close to each other. From Figure 7.9, it is noticed that there is a deviation of the centre and crisp values from 40 cm to 160 cm. After that the centre and crisp values converge. We see that there is a sudden deviation (increase in the uncertain width) of right values in 80 cm to 160 cm and 160 cm to 240 cm. Similarly there is sudden deviation of left values in 40 cm to 120 cm and 120 cm to 200 cm.

Whereas, when only the neutron interaction coefficients ($\Sigma_{r1}, \Sigma_{a2}, \Sigma_{12}$) are fuzzy then the distribution of uncertain thermal group flux are shown in Figure 7.11. It is seen that the left, centre and crisp values are same in the region 0 cm to 40 cm. From 40 cm to 120 cm it may be noticed that there is a sudden deviation of left value. Then from 120 cm to 160 cm the values are same and then there is a deviation. Similarly the right values show deviation throughout the domain except in the region 200 cm to 240 cm, where the right, centre and crisp values are same. Finally, it is investigated that the distribution of centre and crisp thermal group fluxes are almost same along the domain.

Next, we have considered group fission constants ($\nu\Sigma_{f1}, \nu\Sigma_{f2}$) are only fuzzy and the uncertain thermal group fluxes are depicted in Figure 7.13. Here the centre and right values are same in 0 cm to 80 cm and 160 cm to 240 cm. The centre and left values are almost same in 0 cm to 80 cm. In Figure 7.15 it is seen that a continuous fluctuation of the centre and crisp values throughout the domain.

Finally all the parameters are taken as fuzzy and the obtained results are depicted in Figure 7.5, where the crisp and centre values are very close. There is a sudden left and right deviation in between 40 cm to 120 cm and 80 cm to 160 cm respectively. In between 0 cm to 80 cm, the right, centre and crisp values almost overlap.

Further, to check the sensitiveness of the used parameters, the deviation of left and right values from centre and crisp values are shown graphically in Figures 7.15 to 7.18. In view of these four figures, it is concluded that when group fission constants are fuzzy, the system is sensitive. Slight change of the parameter changes drastically the distribution of thermal group fluxes. The width of the uncertain thermal group fluxes increases with larger width values of $\Sigma_{r1}, \Sigma_{a2}, \Sigma_{12}$.

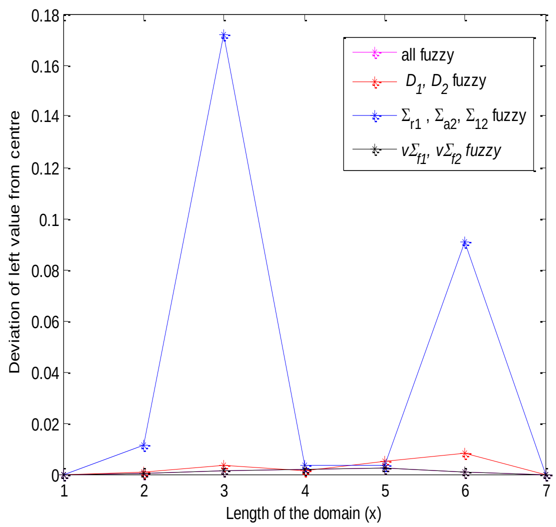

Figure 7.19. Deviation in fast group flux of left value from centre value of the uncertain neutron fluxes

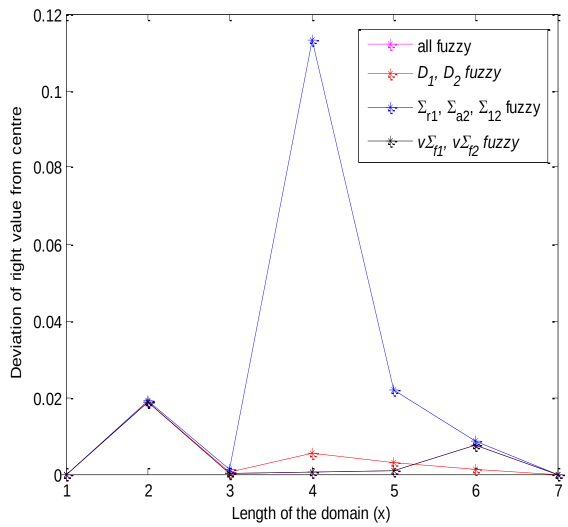

Figure 7.20. Deviation in fast group flux of right value from centre value of the uncertain neutron fluxes

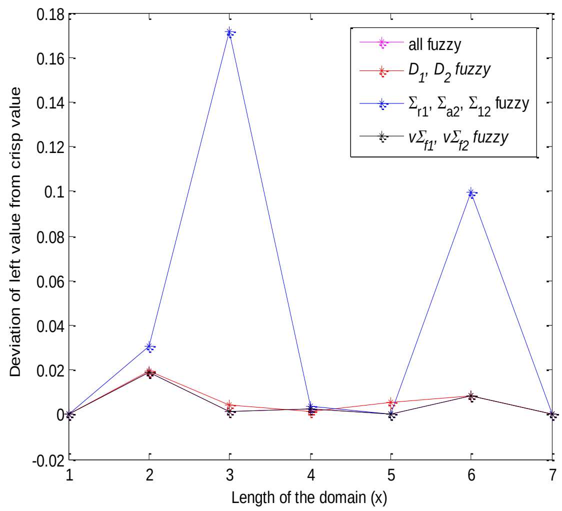

Figure 7.21. Deviation in thermal group flux of left value from crisp value

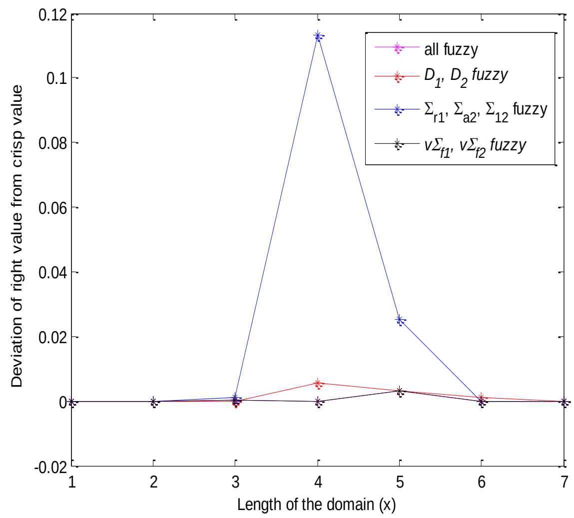

Figure 7.22. Deviation in thermal group flux of right value from crisp value

ii) Fast group neutron fluxes

In Figure 7.10, we have considered only the diffusion coefficients (D_1, D_2) as fuzzy. Here, we observed that the distribution of centre and crisp fast group flux is similar with the case where all parameters are taken as fuzzy. Whereas, the uncertain width maintains a consistency from 80 cm to 160 cm. and there is a deviation of left value distribution in the region 200 cm to 240 cm.

Next we consider only the neutron interaction coefficients ($\Sigma_{r1}, \Sigma_{a2}, \Sigma_{12}$) as fuzzy and obtained results are presented in Figure 7.12. Here we may see that in the region 0 cm to 80 cm, the centre and left values are same and in the same fashion crisp and right distribution of fluxes are same. It is noticed that there is a sharp deviation of right value from 80 cm to 200 cm. Finally from 200 cm to 240 cm the crisp and right distributions are same.

When group fission constants ($v\Sigma_{f1}, v\Sigma_{f2}$) are taken as fuzzy we observed that the right and crisp distributions are same from 0 cm to 120 cm and from 200 cm to 240 cm. Further, it is investigated that the width of the uncertainty decreases in the region 80 cm to 160 cm which may be visualized in Figure 7.14.

Finally, when all the parameters are taken as fuzzy then Figure 7.6 shows that in the region 0 cm to 80 cm the right and crisp values of the fast group flux are almost same. Also from 80 cm to 240 cm, the centre and crisp distribution of fluxes are same. It is seen that there is a constant uncertain width of the fluxes in between 80 cm to 160 cm, whereas, the uncertain width increases widely from 160 cm to 200 cm and then decreases.

To investigate the sensitiveness of the parameters used, the deviation of left and right flux distribution of fast group with the centre and crisp fluxes are depicted graphically in Figures 7.19 to 7.22. It is seen that in both the cases, the case where neutron interaction coefficients $(\Sigma_{r1}, \Sigma_{a2}, \Sigma_{12})$ are fuzzy, system becomes sensitive. Here more the width of uncertainty in these parameters results drastically increase in the width of fast group fluxes.

From above analysis it is seen that when neutron interaction coefficients are fuzzy then the uncertain thermal group and fast group fluxes are more sensitive than the other cases. Here little change in values of neutron interaction coefficients affect more to the distribution of neutron flux and the error or width of uncertainty increases comparing with the other cases.

7.5 Conclusion

This investigation comprises the study of uncertain multigroup neutron diffusion equations. The diffusion coefficients, neutron interaction coefficients and group fission constants are considered as interval/fuzzy. A general model for uncertain multi group neutron diffusion problem has been developed here. The uncertain distributions of fast and thermal neutron populations are also discussed. Various cases are investigated by considering different combinations of fuzzy parameters. A two group bench mark problem has been solved and the sensitivity of the uncertain parameters in the context of fast and thermal neutrons are presented.

Chapter 8

Fuzzy Stochastic Differential Equation

The contents of this chapter have been communicated in the following journals:

1. S. Nayak and S. Chakraverty, Numerical solution of stochastic point kinetic neutron diffusion equation with fuzzy parameters, ***Nuclear Technology***, 2015, (Accepted);
2. S. Nayak and S. Chakraverty, Numerical solution of fuzzy stochastic differential equation, ***Journal of Intelligent & Fuzzy Systems***;
3. S. Nayak and S. Chakraverty, Numerical solution of langevin stochastic differential equation with uncertain parameters, (To be communicated).

Chapter 8
Fuzzy Stochastic Differential Equation

As mentioned earlier, every system posses uncertainty and these uncertainties occur due to the partial knowledge and truth. In general partial knowledge based uncertainties may be handled by probability theory and thuth based uncertainties are operated through possibility theory. Due to the presence of both the uncertainties in the system, this chapter comprises hybridisation of the concept of stochasticity with the fuzzy theory. Accordingly, we have modelled Stochastic Differential Equation (SDE) in terms of fuzzy into Fuzzy Stochastic Differential Equation (FSDE). As such, we have proposed two methods viz. Fuzzy Euler Maruyama Method (FEMM) and Fuzzy Milstein Method (FMM) which are used to investigate various diffusion problems. The details are discussed in this chapter.

8.1 Black-Scholes Stochastic Differential Equation

Here an alternative approach to solve uncertain Stochastic Differential Equation (SDE) has been presented. The uncertainties occurs due to the involved parameters in system and these are considered as Triangular Fuzzy Numbers (TFN). The proposed fuzzy arithmetic in (Nayak & Chakraverty 2013) is used as a tool to handle Fuzzy Stochastic Differential Equation (FSDE). In particular, a system of Ito stochastic differential equations is analysed with fuzzy parameters. Further, exact and Euler Maruyama approximation methods with fuzzy values are demonstrated.

For the sake of completeness initially we have discussed the crisp SDE and it is solved analytically through Ito integral technique. It has also been noted that there was difficulty to handle nontrivial problems using analytical method so we have used numerical method to solve. Further, the same problems are discussed for uncertain cases and corresponding FSDE are solved. The obtained results are shown graphically and the uncertain width of the solution is discussed.

8.1.1 Preliminary

Let us consider a standard stochastic differential equation

$$dX = a(t, X)dt + b(t, X)dW_t \tag{8.1}$$

where Eq. (8.1) is written in differential form.

The integral form of Eq. (8.1) becomes

$$X(t) = X(0) + \int_0^t a(s, y)ds + \int_0^t b(s, y)dW_s \tag{8.2}$$

where the last term in the right hand side of Eq. (8.2) is called Ito integral.

We take $c = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = d$ be a grid of points on an interval $[c, d]$, then Ito integral may be defined in the following limit form

$$\int_c^d f(t)dW_t = \lim_{\Delta t \to 0} \sum_{i=1}^{n} f(t_{i-1})\Delta W_i \tag{8.3}$$

where $\Delta W_i = W_{t_i} - W_{t_{i-1}}$, a step of Brownian motion across the interval.

8.1.2 Analytical Solution of Stochastic Differential Equations (SDE)

Let us consider the Eq. (8.1), which is first solved analytically by using Ito formula.

Ito formula says that if X_t, an Ito process given by

$$dX_t = udt + vdW_t \tag{8.4}$$

Let $g(t, x) \in C^2([0, \infty] \times \Re)$ (i.e. g is twice continuous differentiable on $[0, \infty] \times \Re$). Then $Y_t = g(t, X_t)$ is again Ito process (Oksendal 2003) and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)(dX_t)^2 \tag{8.5}$$

where $(dX_t)^2 = (dX_t).(dX_t)$ is computed as follows

$$\begin{aligned} &dt.dt = dt.dW_t = dW_t.dt = 0; \\ &dW_t.dW_t = dt. \end{aligned}$$

Example 8.1.1

Let us consider a stochastic differential equation (Malinowski & Michta 2011)

$$\begin{cases} \dfrac{dX_t}{dt} = a_t X_t, \\ X(0, x) = X_0 \end{cases} \tag{8.6}$$

where $a_t = r_t + \alpha W_t$

W_t and α are noise and constant respectively.

Eq. (8.6) may be written as

$$\frac{dX_t}{dt} = (r_t + \alpha W_t)X_t$$
$$= r_t X_t + \alpha W_t X_t$$
$$\Rightarrow dX_t = r_t X_t dt + \alpha X_t dB_t \quad (\because W_t.dt = dB_t)$$
$$\Rightarrow \frac{dX_t}{X_t} = r_t dt + \alpha dB_t$$
$$\Rightarrow \int_0^t \frac{dX_s}{X_s} = r_t t + \alpha B_t$$

Using Ito formula for the function $g(t,x) = \ln x$ (Oksendal 2003), we get the following

$$d(\ln X_t) = \frac{1}{X_t} dX_t - \frac{1}{2}\left(\frac{1}{X_t^{\,2}}\right)(dX_t)^2$$
$$= \frac{1}{X_t} dX_t - \frac{1}{2}\left(\frac{1}{X_t^{\,2}}\right)\alpha^2 X_t^{\,2} dt$$
$$= \frac{1}{X_t} dX_t - \frac{1}{2}\alpha^2 dt$$

Integrating the above we get,

$$\int_0^t d(\ln X_t) = \int \frac{dX_t}{X_t} - \int \frac{1}{2}\alpha^2 dt$$
$$\Rightarrow \ln X_t - \ln X_0 = \int r_t dt + \alpha dB_t - \frac{1}{2}\alpha^2 t$$
$$\Rightarrow \ln \frac{X_t}{X_0} = \left(r_t - \frac{1}{2}\alpha^2\right)t + \alpha B_t$$
$$\Rightarrow X_t = X_0 e^{\left(r_t - \frac{1}{2}\alpha^2\right)t + \alpha B_t}$$

Here we found that except some standard problems exact method may not be applicable for others. Hence we need numerical treatment to handle non trivial problems and this has been discussed in the following sections.

8.1.3 Solution of Fuzzy Stochastic Differential Equations (FSDE)

Let us consider a SDE with fuzzy parameters then Eq. (8.1) may be written as

$$d[\underline{X}(\alpha), \overline{X}(\alpha)] = [\underline{a}(\alpha), \overline{a}(\alpha)]dt + [\underline{b}(\alpha), \overline{b}(\alpha)]dW_t \tag{8.7}$$

Now Eq. (8.7) is solved by exact and numerical methods respectively.

Using limit method (Chakraverty & Nayak 2013), the FSDE (8.7) in modified limit form may be represented as follows

$$d\{\lim_{s\to\infty} X(\alpha), \lim_{s\to 1} X(\alpha)\} = \{\lim_{s\to\infty} a(\alpha), \lim_{s\to 1} a(\alpha)\}dt + \{\lim_{s\to\infty} b(\alpha), \lim_{s\to 1} b(\alpha)\}dW_t \qquad (8.8)$$

where

$\tilde{X}(\alpha) = \underline{X}(\alpha) + \dfrac{\overline{X}(\alpha) - \underline{X}(\alpha)}{s}$, $\tilde{a}(\alpha) = \underline{a}(\alpha) + \dfrac{\overline{a}(\alpha) - \underline{a}(\alpha)}{s}$ and $\tilde{b}(\alpha) = \underline{b}(\alpha) + \dfrac{\overline{b}(\alpha) - \underline{b}(\alpha)}{s}$.

Initially for the exact (or crisp) case, we take the crisp representation of $\tilde{X}(\alpha), \tilde{a}(\alpha), \tilde{b}(\alpha)$ and use Ito integral to solve the problem.

Now if we apply the above discussed fuzzy concept for Euler-Maruyam method (Chapter 3), then we get

$$\begin{aligned} &\tilde{X}_0(\alpha) = \tilde{w}_0(\alpha) \\ &\tilde{w}_{i+1}(\alpha) = \tilde{w}_i(\alpha) + \tilde{a}(t_i, w_i, \alpha)\Delta t_{i+1} + \tilde{b}(t_i, w_i, \alpha)\Delta W_i \end{aligned} \qquad (8.9)$$

where

$\tilde{X}_0(\alpha) = \underline{X_0}(\alpha) + \dfrac{\overline{X}_0(\alpha) - \underline{X_0}(\alpha)}{s}$, $\tilde{w}_0(\alpha) = \underline{w_0}(\alpha) + \dfrac{\overline{w}_0(\alpha) - \underline{w_0}(\alpha)}{s}$,

$\tilde{w}_{i+1}(\alpha) = \underline{w_{i+1}}(\alpha) + \dfrac{\overline{w}_{i+1}(\alpha) - \underline{w_{i+1}}(\alpha)}{s}$, $\tilde{a}(t_i, w_i, \alpha) = \underline{a}(t_i, w_i, \alpha) + \dfrac{\overline{a}(t_i, w_i, \alpha) - \underline{a}(t_i, w_i, \alpha)}{s}$

and $\tilde{b}(t_i, w_i, \alpha) = \underline{b}(t_i, w_i, \alpha) + \dfrac{\overline{b}(t_i, w_i, \alpha) - \underline{b}(t_i, w_i, \alpha)}{s}$.

Applying $\lim_{s\to\infty}$ and $\lim_{s\to 1}$ on the solution we get the left and right bound. Whereas, we obtain various solution set by considering different values of $\alpha \in [0,1]$. It is noticed that sometimes we get weak solutions i.e. the left and right bound solutions overlaps or intersect each other and this occurs due to the randomness of the system. This may easily be observed from the following example problems.

8.1.4 Example Problems

In this section we have considered two example problems and taken parameters as fuzzy. Initially the problem is studied for crisp parameters for both the exact and numerical methods and then the fuzzy parameter are incorporated.

Example 8.1.2

Consider Black Scholes stochastic differential equation (Black & Scholes 1973).

The crisp Euler-Maruyama approximation for this SDE is as follows

$$w_0 = X_0$$
$$w_{i+1} = w_i + \mu w_i \Delta t_i + \sigma w_i \Delta W_i \tag{8.10}$$

where, the values of involved parameters are given in Table 8.1.

Table 8.1. Crisp and fuzzy values of the involved parameters

Parameters	crisp	TFN
μ	0.75	[0.65, 0.75, 0.85]
σ	0.30	[0.25, 0.30, 0.35]

Initially the Black Scholes SDE has been solved for crisp parameter and then fuzzy parameters are considered for investigation. Here we compute a discretized Brownian path over [0, 1] with $\delta t = 2^{-8}$ and the obtained solution is plotted with a solid magenta line in Figure 8.1. We then apply Euler Maruyama (EM) method using a step size $\Delta t = R\delta t$, with R=4 and obtain the solution which is presented in Figure 8.1 with blue line. Further it is seen that taking smaller value of R for 4, 3 and 2 we get the endpoint errors 0.0442, 0.0216 and 0.0101 respectively.

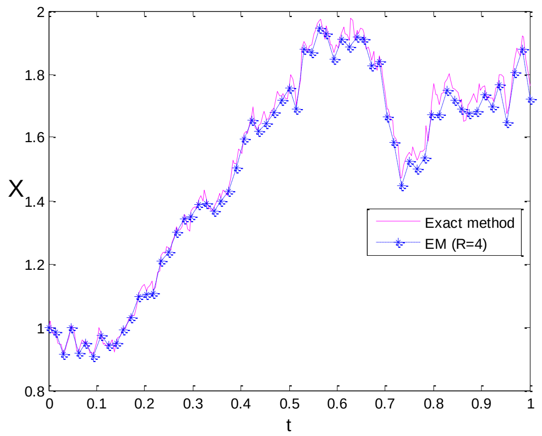

Figure 8.1. Solution of Black Scholes SDE when parameters are crisp

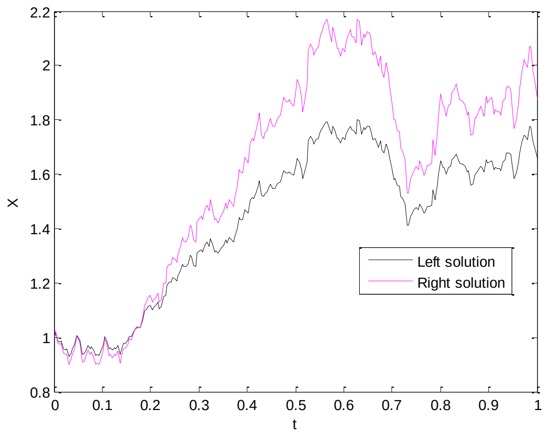

Figure 8.2. Exact solution of Black Scholes SDE when parameters are fuzzy

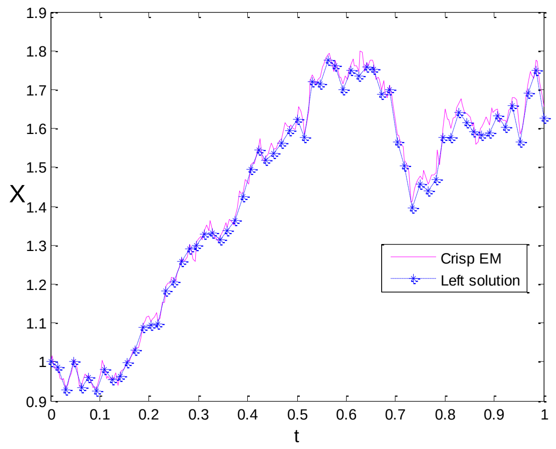

Figure 8.3. Crisp Euler Maruyama solution of Black Scholes SDE and the left uncertain bound solution

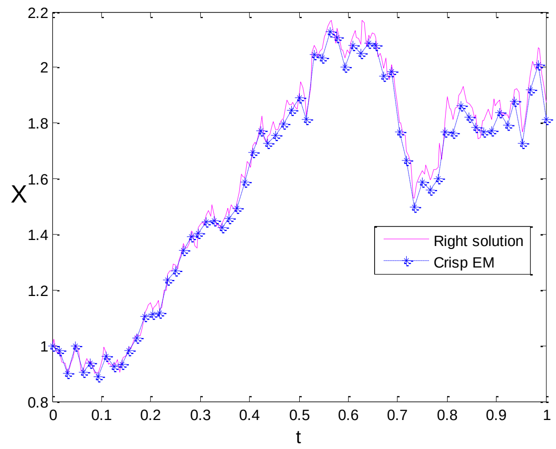

Figure 8.4. Crisp Euler Maruyama solution of Black Scholes SDE and the right uncertain bound solutions

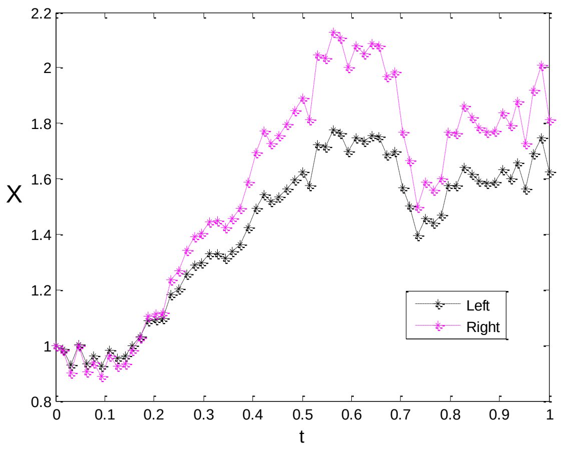

Figure 8.5. Euler Maruyama solution of Black Scholes SDE when parameters are fuzzy

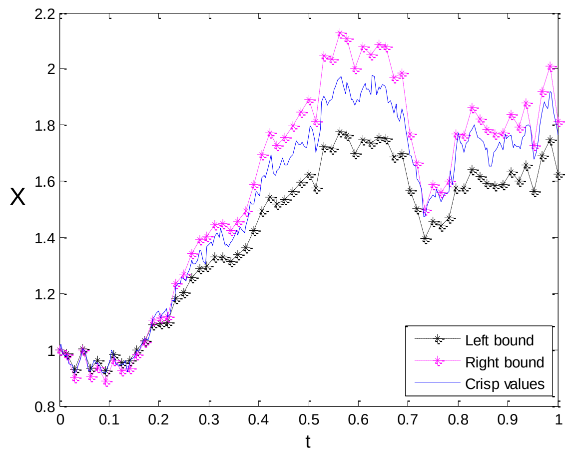

Figure 8.6. Euler Maruyama solution of Black Scholes SDE when parameters are fuzzy with the exact solution

Now, drift (μ) and diffusion (σ) coefficients are taken as TFN which are given in Table 8.1. The exact method is used to obtain the solution which is depicted in Figure 8.2. Here the black and magenta solid line represents the left and right bound of the uncertainty. Next the left and right values of the uncertainty are plotted with the exact solution in Figures 8.3 and 8.4 respectively. Then EM method is used to solve the uncertain SDE and results are graphically depicted in Figure 8.5, where black and magenta line represents the left and right bound of the uncertain solutions. The region covered in between the left and right bound is the uncertain solution set of the Black Scholes SDE. In Figure 8.6, we have given the left and right bound which represent the uncertain solution of the Black Scholes SDE along with the crisp solution and we found that the exact solution lies within the region covered by the left and right solutions. Further, it is found that there are some problems where it may be difficult to find the exact solution and in this case we take the help of Euler-Maruyama method which is discussed in the next example.

Example 8.1.3

The SDE of Langevin equation is

$$dX(t) = -\mu X(t)dt + \sigma dW_t \tag{8.11}$$

where μ and σ are positive constants.

The Euler Maruyama approximation for Eq. (8.11) is as follows

$$\begin{aligned} w_0 &= X_0 \\ w_{i+1} &= w_i - \mu w_i \Delta t_i + \sigma \Delta W_i \end{aligned} \tag{8.12}$$

The values for used parameters in Eq. (8.8) are given in the following Table 8.2.

Table 8.2. Crisp and fuzzy values of the used parameters

Parameters	crisp	TFN
μ	10	[8, 10, 12]
σ	1	[0.5, 1, 1.5]

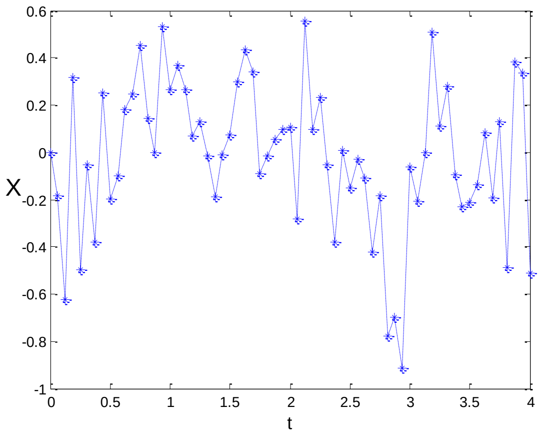

Figure 8.7. Euler Maruyama solution of Langevin SDE when parameters are fuzzy

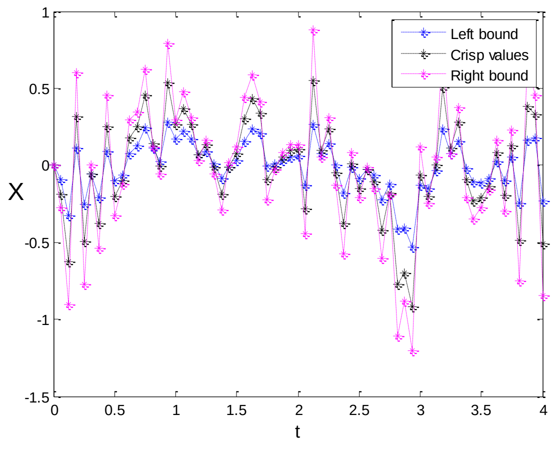

Figure 8.8. Euler Maruyama solution of Langevin SDE when parameters are fuzzy with the crisp solution

In Figure 8.9, we have given a plot for the solution of Langevin SDE when parameters are crisp. Whereas, the solution for Langevin SDE is presented in Figure 8.8 where the parameters are taken as fuzzy. The left and right bound solutions are shown in blue and magenta colour respectively.

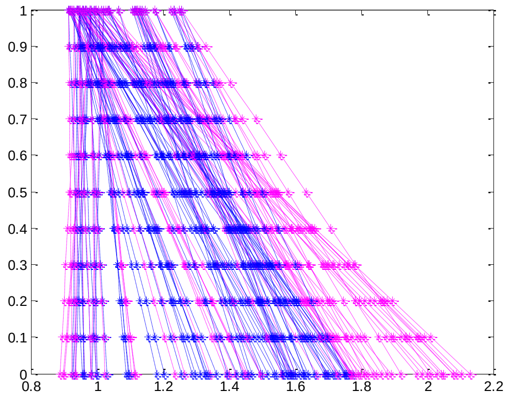

Figure 8.9. Fuzzy plot of Euler Maruyama solution of Black Scholes SDE when parameters are TFN (Example 8.1.2)

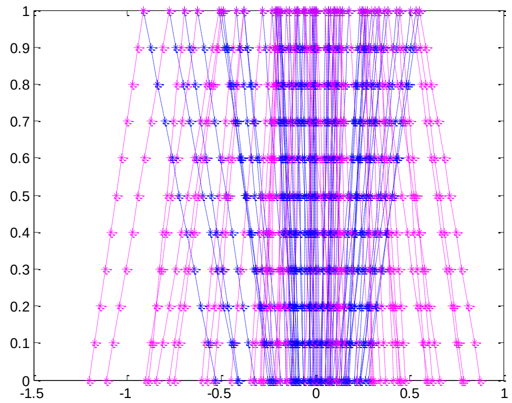

Figure 8.10. Fuzzy plot of Euler Maruyama solution of Langevin SDE when parameters are TFN (Example 8.1.3)

For better visualization of uncertain distribution of Euler Maruyama approximation results, fuzzy plots are represented in Figures 8.9 and 8.10 for examples 8.1.2 and 8.1.3 respectively. Here one may also be seen that the uncertain widths are randomly varying. It may be noted that if the uncertainty of the parameter changes, the uncertain width of the solution sets vary accordingly.

8.2 Langevin Stochastic Differential Equation

Here two different approaches viz. Fuzzy Euler Maruyama Method (FEMM) and Fuzzy Milstein Method (FMM) are presented for solving uncertain Langevin Stochastic Differential Equations (SDE). The uncertainties are taken in initial conditions as well as associated parameters in term of Triangular Fuzzy Numbers (TFN). The limit method (Chapter 3) for fuzzy arithmetic has been used as a tool to handle Fuzzy Stochastic Differential Equation (FSDE).

8.2.1 Solution of Fuzzy Stochastic Differential Equations (FSDE)

Let us consider a SDE with fuzzy parameters then Eq. (8.8) may be classified into following three categories.

Case 1

For the first category we have considered the initial conditions of the SDE as uncertain viz. fuzzy.

As such, we consider Eq. (8.8) with initial condition as fuzzy and then we have

$$\begin{cases} dX(t) = a(t, X)dt + b(t, X)dW_t \\ \tilde{X}(0) = [\underline{X}_0(\alpha), \overline{X}_0(\alpha)] \end{cases} \tag{8.13}$$

Now if we apply the above discussed fuzzy concept for Euler-Maruyam method, then Eq. (8.13) may be represented in the following way

$$\begin{aligned} &\tilde{X}_0(\alpha) = \tilde{w}_0(\alpha) \\ &\tilde{w}_{i+1}(\alpha) = \tilde{w}_i(\alpha) + \tilde{a}(t_i, w_i)\Delta t_{i+1} + \tilde{b}(t_i, w_i)\Delta W_i \end{aligned} \tag{8.14}$$

where

$$\tilde{X}_0(\alpha) = \underline{X}_0(\alpha) + \frac{\overline{X}_0(\alpha) - \underline{X}_0(\alpha)}{s},\ \tilde{w}_0(\alpha) = \underline{w}_0(\alpha) + \frac{\overline{w}_0(\alpha) - \underline{w}_0(\alpha)}{s}.$$

Further it is noticed that due to the fuzziness of the initial value we get series of approximated fuzzy solutions and these are

$$\tilde{w}_{i+1}(\alpha) = \underline{w}_{i+1}(\alpha) + \frac{\overline{w}_{i+1}(\alpha) - \underline{w}_{i+1}(\alpha)}{s},\ i \geq 0$$

When Milstein method is used to handle this situation (Eq. (8.13)) then the approximation scheme may be represented in the following manner

$$\begin{aligned} &\tilde{X}_0(\alpha) = \tilde{w}_0(\alpha) \\ &\tilde{w}_{i+1}(\alpha) = \tilde{w}_i(\alpha) + \tilde{a}(t_i, w_i)\Delta t_{i+1} + \tilde{b}(t_i, w_i)\Delta W_i + \frac{1}{2}\tilde{b}(t_i, w_i)\frac{\partial \tilde{b}}{\partial x}(t_i, w_i)(\Delta W_i^2 - \Delta t_i) \end{aligned} \tag{8.15}$$

Case 2

In this case we assume the involve parameters (or the coefficients) only as fuzzy then the modified form of Eq. (8.8) may be written as

$$\begin{cases} d\tilde{X}(t) = [\underline{a}(t, X, \alpha), \overline{a}(t, X, \alpha)]dt + [\underline{b}(t, X, \alpha), \overline{b}(t, X, \alpha)]dW_t \\ X(0) = X_0 \end{cases} \tag{8.16}$$

By applying the above discussed fuzzy concept for Euler-Maruyam method, Eq. (8.16) may be represented as

$$\begin{aligned} &X_0 = w_0 \\ &\tilde{w}_{i+1} = \tilde{w}_i + \tilde{a}(t_i, w_i, \alpha)\Delta t_{i+1} + \tilde{b}(t_i, w_i, \alpha)\Delta W_i \end{aligned} \tag{8.17}$$

where

$$\tilde{a}(t_i, w_i, \alpha) = \underline{a}(t_i, w_i, \alpha) + \frac{\overline{a}(t_i, w_i, \alpha) - \underline{a}(t_i, w_i, \alpha)}{s}$$

and $\tilde{b}(t_i, w_i, \alpha) = \underline{b}(t_i, w_i, \alpha) + \dfrac{\overline{b}(t_i, w_i, \alpha) - \underline{b}(t_i, w_i, \alpha)}{s}$.

Whereas, for Milstein method the approximation scheme for Eq. (8.16) may be written as

$$X_0 = w_0$$
$$\tilde{w}_{i+1} = \tilde{w}_i + \tilde{a}(t_i, w_i, \alpha)\Delta t_{i+1} + \tilde{b}(t_i, w_i, \alpha)\Delta W_i + \frac{1}{2}\tilde{b}(t_i, w_i, \alpha)\frac{\partial b}{\partial x}(t_i, w_i, \alpha)(\Delta W_i^2 - \Delta t_i) \quad (8.18)$$

It may be pointed out that as the parameters are taken as fuzzy we get series of approximated fuzzy solutions in α -cut form and these are $\tilde{w}_{i+1}(\alpha) = \underline{w_{i+1}}(\alpha) + \dfrac{\overline{w}_{i+1}(\alpha) - \underline{w_{i+1}}(\alpha)}{s}$.

Case 3

Here, initial condition as well as involved parameters both is considered as fuzzy and Eq. (8.8) may be represented in the following manner

$$\begin{cases} d\tilde{X}(t) = [\underline{a}(t, X, \alpha), \overline{a}(t, X, \alpha)]dt + [\underline{b}(t, X, \alpha), \overline{b}(t, X, \alpha)]dW_t \\ \tilde{X}(0) = [\underline{X}_0(\alpha), \overline{X}_0(\alpha)] \end{cases} \quad (8.19)$$

Applying the above discussed fuzzy concept for Euler-Maruyam method, Eq. (8.19) may now be represented as

$$\begin{aligned} &\tilde{X}_0(\alpha) = \tilde{w}_0(\alpha) \\ &\tilde{w}_{i+1}(\alpha) = \tilde{w}_i(\alpha) + \tilde{a}(t_i, w_i, \alpha)\Delta t_{i+1} + \tilde{b}(t_i, w_i, \alpha)\Delta W_i \end{aligned} \quad (8.20)$$

and for Milstein method, the approximation scheme becomes

$$\begin{aligned} &\tilde{X}_0(\alpha) = \tilde{w}_0(\alpha) \\ &\tilde{w}_{i+1}(\alpha) = \tilde{w}_i(\alpha) + \tilde{a}(t_i, w_i, \alpha)\Delta t_{i+1} + \tilde{b}(t_i, w_i, \alpha)\Delta W_i \\ &\qquad + \frac{1}{2}\tilde{b}(t_i, w_i, \alpha)\frac{\partial b}{\partial x}(t_i, w_i, \alpha)(\Delta W_i^2 - \Delta t_i) \end{aligned} \quad (8.21)$$

The above method is demonstrated and are compared with the special cases through an example problem which is discussed in the next subsection.

8.2.2 Example Problem

Let us consider SDE of Langevin equation

$$dX(t) = -\mu X(t)dt + \sigma X dW_t \quad (8.22)$$

where μ and σ are positive constants.

Table 8.3. Crisp and fuzzy values of the used parameters

Parameters	crisp	TFN
X_0	1	[0.5, 1, 1.5]
μ	15	[10, 15, 20]
σ	1	[0.5, 1, 1.5]

The parameters for Eq. (8.22) in term of crisp and fuzzy are given in Table 8.3.

Case 1

In this case only initial condition is fuzzy. As mentioned earlier, Euler Maruyama and Milstein methods are used to handle the problem. The investigated results are depicted in Figures 8.11 to 8.14. FEMM solutions are given in Figures 8.11 and 8.12 for alpha=0.5 and 0 (interval). Similarly, Figures 8.13 and 8.14 depicts the FMM results for alpha=0.5 and 0 (interval).

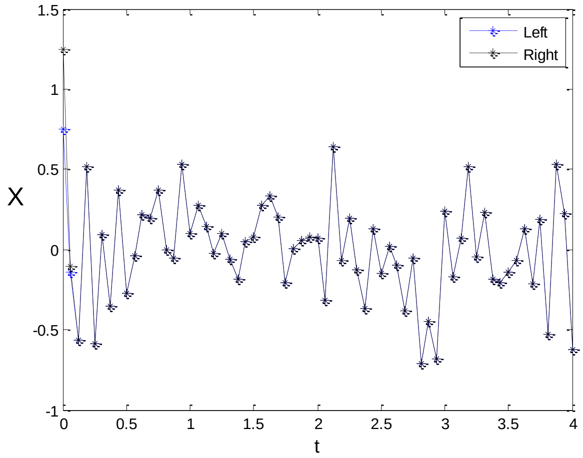

Figure 8.11. Interval solutions using FEMM at $\alpha = 0.5$

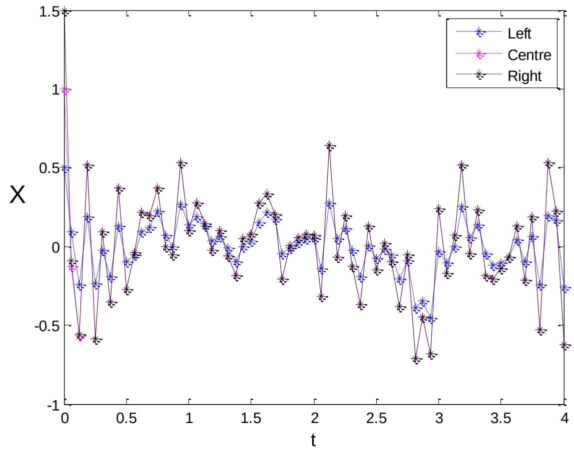

Figure 8.12. Fuzzy solutions using FEMM

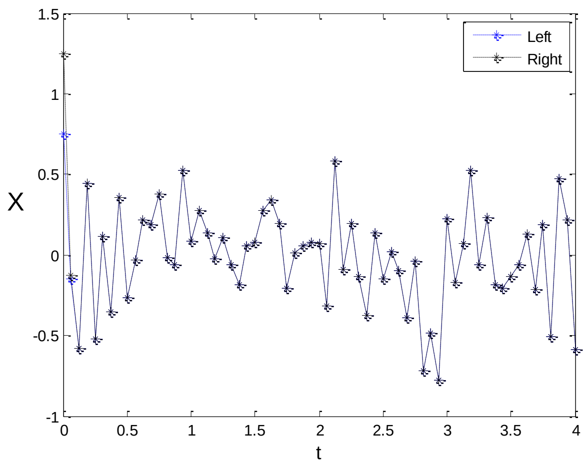

Figure 8.13. Interval solutions using FMM at $\alpha = 0.5$

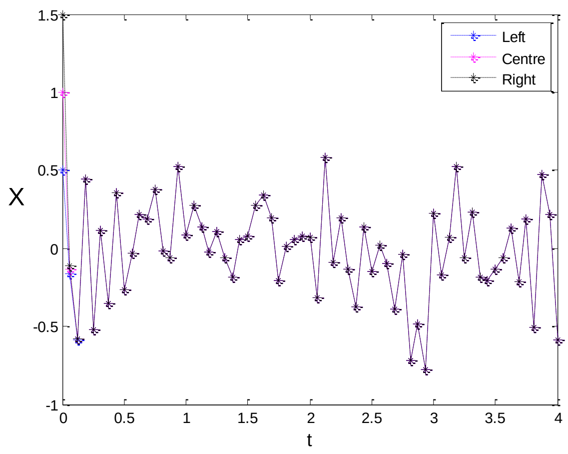

Figure 8.14. Fuzzy solutions using FMM

Case 2

Here, the involved parameters (except initial condition) are considered as fuzzy and the solutions for FEMM and FMM are shown in Figures 8.15 to 8.18 for the values of alpha as mentioned in Case 1.

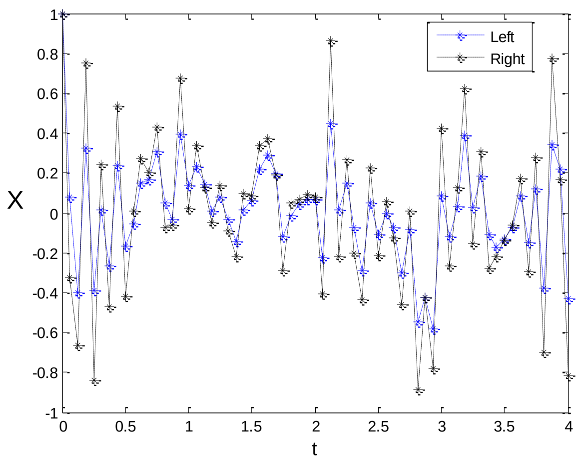

Figure 8.15. Interval solutions using FEMM at $\alpha = 0.5$

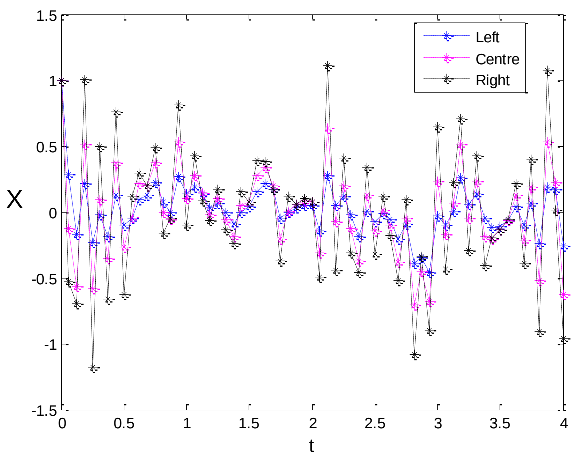

Figure 8.16. Fuzzy solutions using FEMM

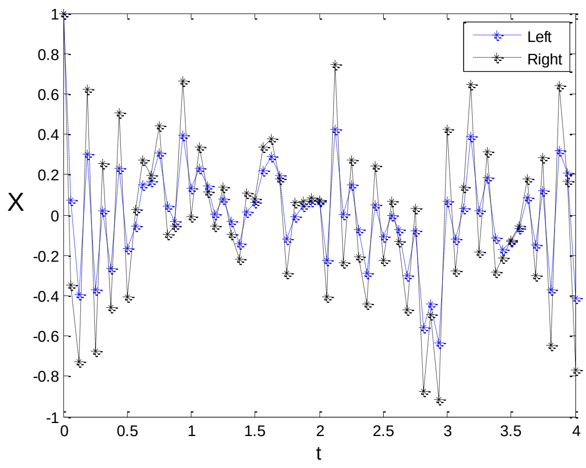

Figure 8.17. Interval solutions using FMM at $\alpha = 0.5$

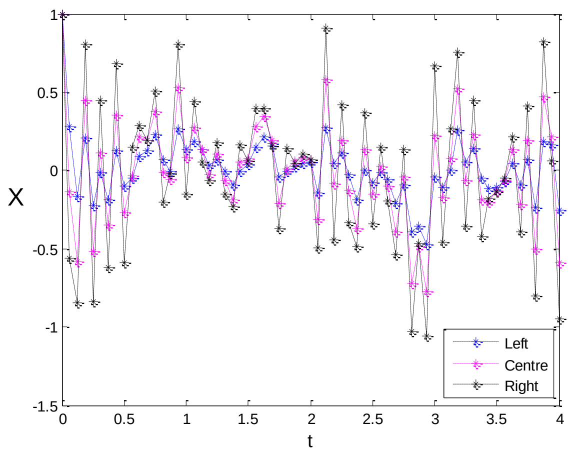

Figure 8.18. Fuzzy solutions using FMM

Case 3

Finally, both the initial condition and parameters are taken as fuzzy and the problem is investigated using Fuzzy Euler Maruyama and Milstein methods. The solutions may be visualized from Figures 8.19 to 8.22 for same values of α as in earlier cases.

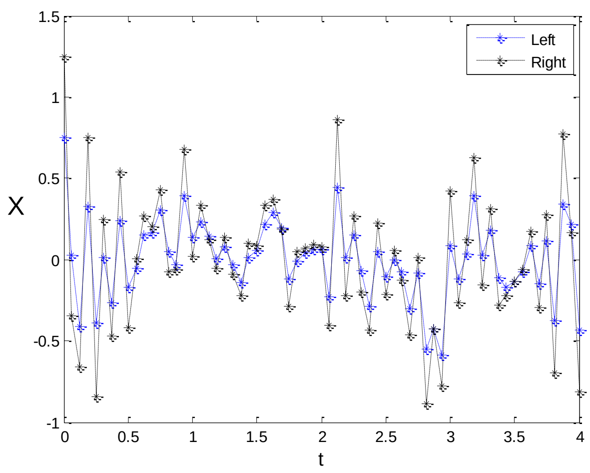

Figure 8.19. Interval solutions using FEMM at $\alpha = 0.5$

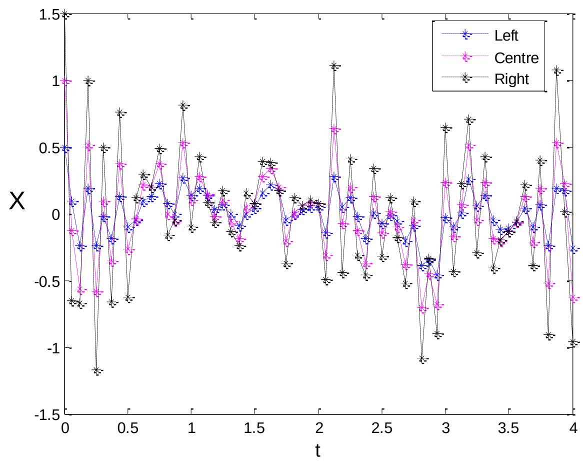

Figure 8.20. Fuzzy solutions using FEMM

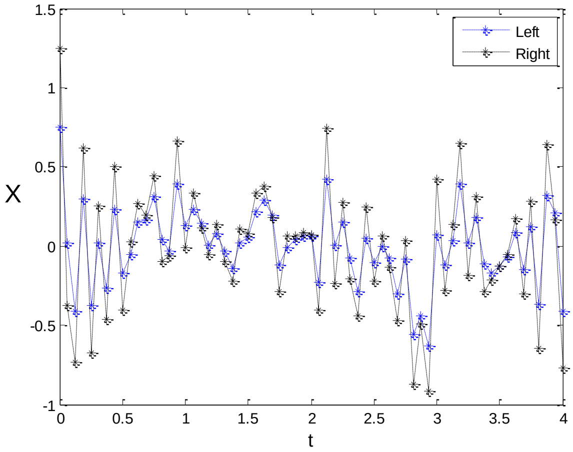

Figure 8.21. Interval solutions using FMM at $\alpha = 0.5$

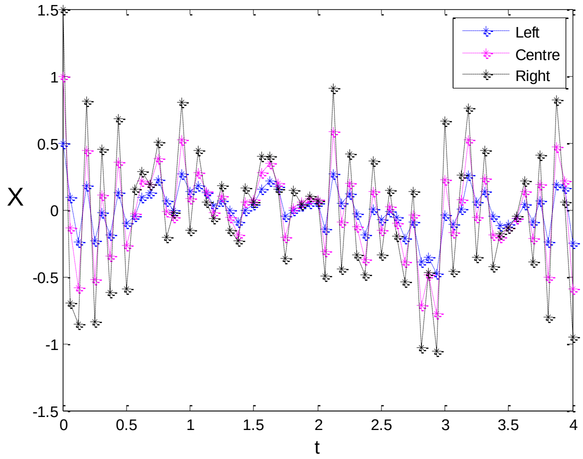

Figure 8.22. Fuzzy solutions using FMM

Above results in digital form are also incorporated in Table 8.4. Here the left, centre and right values of the TFNs are given for different cases. Further, the values of X at different time $(t = 1, 2, 3, 4)$ denoted as X (1), X (2), X (3) and X (4) are presented in this table for both the Fuzzy Euler Maruyama Method (FEMM) and Fuzzy Milstein Method (FMM).

Table 8.4. Fuzzy solution of the problem for different cases

		Crisp		Left		Centre		Right	
		EMM	MM	FEMM	FMM	FEMM	FMM	FEMM	FMM
Case 1	X(1)	0.1003	0.0847	0.1337	0.0847	0.1003	0.0847	0.1003	0.0847
	X(2)	0.0732	0.0708	0.0523	0.0708	0.0732	0.0708	0.0732	0.0708
	X(3)	0.2367	0.2250	-0.0319	0.2250	0.2367	0.2250	0.2367	0.2250
	X(4)	-0.6246	-0.5883	-0.2565	-0.5883	-0.6246	-0.5883	-0.6246	-0.5883
Case 2	X(1)	0.1003	0.0847	0.1337	0.1316	0.1003	0.0847	-0.1029	-0.1525
	X(2)	0.0732	0.0708	0.0523	0.0519	0.0732	0.0708	0.0770	0.0700
	X(3)	0.2367	0.2250	-0.0319	-0.0399	0.2367	0.2250	0.6432	0.6647
	X(4)	-0.6246	-0.5883	-0.2565	-0.2507	-0.6246	-0.5883	-0.9620	-0.9499
Case 3	X(1)	0.1003	0.0847	0.1003	0.1316	0.1003	0.0847	-0.1029	-0.1525
	X(2)	0.0732	0.0708	0.0732	0.0519	0.0732	0.0708	0.0770	0.0700
	X(3)	0.2367	0.2250	0.2367	-0.0399	0.2367	0.2250	0.6432	0.6647
	X(4)	-0.6246	-0.5883	-0.6246	-0.2507	-0.6246	-0.5883	-0.9620	-0.9499

The sensitiveness of the solution set are studied in term of the uncertain widths. Here we have considered the fuzzy solutions and widths of X at different time $(t = 1, 2, 3, 4)$ which are encrypted in Table 8.5. The uncertain widths for both FEMM and FMM are given in this Table and it may be observed that FMM gives less width.

Table 8.5. Width of the solutions at $\alpha = 0$ using FEMM and FMM

		Width	
		Fuzzy Euler Maruyama Method (FEMM)	Fuzzy Milstein Method (FMM)
Case 1	X(1)	0.0334	0
	X(2)	0.0209	0
	X(3)	0.2686	0
	X(4)	0.3681	0
Case 2	X(1)	0.2366	0.2841
	X(2)	0.0247	0.0189
	X(3)	0.6751	0.7046
	X(4)	0.7055	0.6992
Case 3	X(1)	0.2032	0.2841
	X(2)	0.0038	0.0189
	X(3)	0.4065	0.7046
	X(4)	0.3374	0.6992

Next, the uncertain solutions (X at $t = 2$) are plotted in term of TFN for various cases in Figures 8.23 to 8.25.

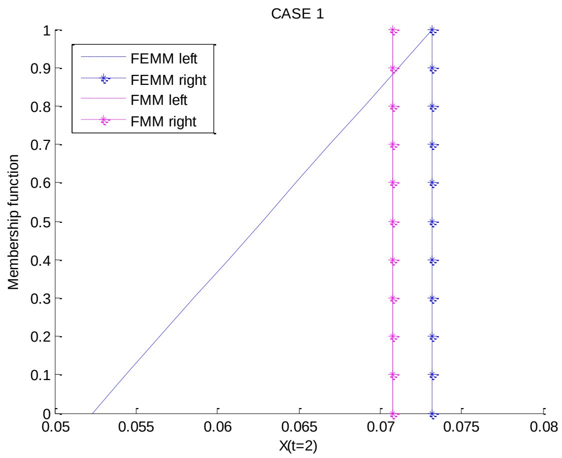

Figure 8.23. Fuzzy plot for case 1 at $X(t = 2)$

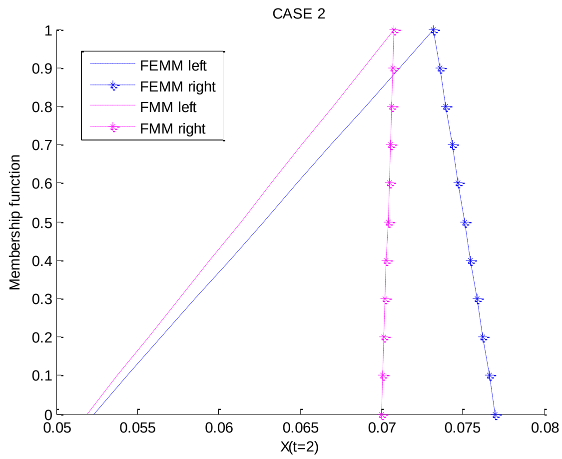

Figure 8.24. Fuzzy plot for case 2 at $X(t = 2)$

In Figure 8.23, it is seen that the solution obtained by using FMM gives a straight line TFN which is parallel to membership function axis. Again, it has also been seen that the non-increasing left continuous function of the obtained TFN by using FFEM becomes parallel to the membership function axis. Whereas, in Figure 8.24, the left non-decreasing function values of the resultant TFNs are more approximate as compared to the left non-increasing function values.

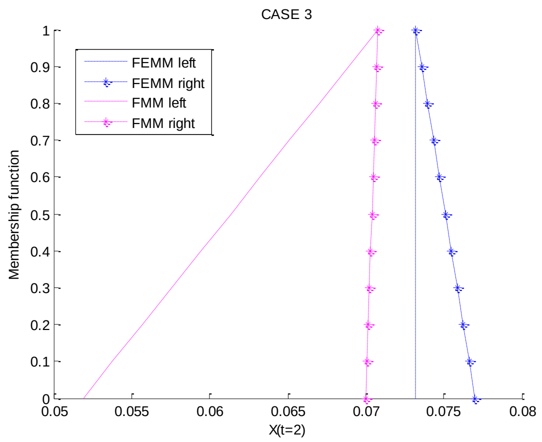

Figure 8.25. Fuzzy plot for case 3 at $X(t=2)$

For case 3, Figure 8.25 gives straight line for the non-increasing and non-decreasing left continuous function using FEMM and FEM respectively which are parallel to membership function axis.

8.3 Point Kinetic Neutron Diffusion Equation

In this section, the concept of fuzziness along with stochastic behaviour of point kinetic neutron diffusion equation has been modelled. The fuzzy stochastic model is investigated numerically by extending Euler-Maruyama Method (EMM) to fuzzy form. Uncertain neutron density and delayed neutron population are obtained and compared with Monte Carlo and stochastic Principal Component Analysis (PCA) solutions. Various combinations of fuzzy parameters are considered and the uncertain neutron density and delayed neutron population are obtained in different cases. Finally, the sensitivity of these fuzzy parameters with stochastic behaviour has been investigated.

8.3.1 Stochastic Point Kinetic Model with Fuzzy Parameters

The point kinetic equation has been modelled in term of stochastic (Hayes & Allen 2005) by considering birth and death process of the neutron and precursor population. The coupled deterministic time dependent equations for the neutron density and the delayed neutron precursors may be represented as

$$\frac{\partial N}{\partial t} = Dv\nabla^2 N - (\Sigma_a - \Sigma_f)vN + [(1-\beta)k_\infty \Sigma_a - \Sigma_f]vN + \sum_i \lambda_i C_i + S_0 \qquad (8.23)$$

$$\frac{\partial C_i}{\partial t} = \beta_i k_\infty \sum_a vN - \lambda_i C_i \tag{8.24}$$

for $i = 1, 2, ..., m$.

where $N = N(r,t)$ is neutron density at position r at time t, v is velocity of the neutron, $\sum_f$ is neutron fission cross section, D is diffusion coefficient, $\sum_a$ is absorption coefficient, $\beta = \sum_{i=1}^{m} \beta_i$ is delayed neutron fraction, $1-\beta$ is prompt neutron fraction, k_∞ is infinite medium neutron reproduction factor, λ_i is delay constant, $C_i = C_i(r,t)$ density of the 'i' th type of precursors at position r at time t, S_0 is extraneous neutron source, $Dv\nabla^2 N$ is diffusion term of the neutrons, $(\sum_a - \sum_f)$ is capture cross section, $[(1-\beta)k_\infty \sum_a - \sum_f]vN$ is prompt neutron contribution to the source and $\sum_{i=1}^{m} \lambda_i C_i$ is rate of transformations from the neutron precursors to the neutron population.

Here, we have considered uncertain parameters viz. delayed neutron fraction, source and initial condition as fuzzy in the stochastic differential equations (Eqs. (8.23) and (8.24)). The fuzzy delayed neutron fraction $\tilde{\beta}$ in α-cut form may be written as $\tilde{\beta} = [\underline{\beta}(\alpha), \overline{\beta}(\alpha)]$.

The above fuzzy parameters are introduced in Eqs. (8.23) and (8.24), which give

$$\frac{\partial \tilde{N}}{\partial t} = Dv\nabla^2 \tilde{N} - (\sum_a - \sum_f)v\tilde{N} + [(1-\tilde{\beta})k_\infty \sum_a - \sum_f]v\tilde{N} + \sum_i \lambda_i C_i + S_0 \tag{8.25}$$

$$\frac{\partial \tilde{C}_i}{\partial t} = \tilde{\beta}_i k_\infty \sum_a v\tilde{N} - \lambda_i \tilde{C}_i \tag{8.26}$$

for $i = 1, 2, ..., m$.

where '~' represents fuzzy numbers.

The neutron captures are considered as deaths whereas the fission process is taken as a pure birth process, where $\nu - 1$ neutrons born in each fission along with a precursor contribution. The number of new neutrons born in each fission is $(1-\tilde{\beta})\nu - 1$. Let us consider $\tilde{N} = \tilde{f}(r)\tilde{n}(t)$ and $\tilde{C}_i = \tilde{g}_i(r)\tilde{c}_i(t)$ where it is assumed that $\tilde{N}$ and $\tilde{C}_i$ are separable in time and space. Proceeding further using (Hayes & Allen 2005; Hetrick 1971), we get the following uncertain point kinetic equation

$$\frac{d\tilde{n}}{dt} = -\left[\frac{-\rho+1-\alpha}{l}\right]\tilde{n} + \left[\frac{1-\alpha-\tilde{\beta}}{l}\right]\tilde{n} + \sum_{i=1}^{m}\lambda_i\tilde{c}_i + \tilde{q} \tag{8.27}$$

$$\frac{d\tilde{c}_i}{dt} = \frac{\tilde{\beta}_i}{l}\tilde{n} - \lambda_i\tilde{c}_i, \qquad i = 1, 2, ..., m \tag{8.28}$$

where, $\tilde{n}(t)$ is total number of uncertain neutrons, $\tilde{c}_i(t)$ is total number of uncertain precursors of the 'i' th type at time t, the reactivity $\rho = \frac{\sum_a(1-k_\infty)+D\nabla^2}{\sum_a + D\nabla^2}$, the number of neutrons per fission $\alpha = \frac{\sum_f}{\sum_a k_\infty} \approx \frac{1}{\nu}$, $l = \frac{1}{\nu\sum_a k_\infty}$ and $\tilde{q} = \tilde{q}(t) = \frac{S_0(r,t)}{\tilde{f}(r)}$.

Applying the procedure of (Hayes & Allen 2005), we obtain the following Ito stochastic differential equation of system for one precursor

$$\frac{d}{dt}\begin{bmatrix}\tilde{n}\\ \tilde{c}_1\end{bmatrix} = \tilde{\hat{A}}\begin{bmatrix}\tilde{n}\\ \tilde{c}_1\end{bmatrix} + \begin{bmatrix}\tilde{q}\\ 0\end{bmatrix} + \tilde{\hat{B}}^{\frac{1}{2}}\frac{d\vec{W}}{dt} \tag{8.29}$$

where $\tilde{\hat{A}} = \begin{bmatrix}\frac{\rho-\tilde{\beta}}{l} & \lambda_1\\ \frac{\tilde{\beta}_1}{l} & -\lambda_1\end{bmatrix}$,

$\tilde{\hat{B}} = \begin{bmatrix}\gamma n + \lambda_1 c_1 + \tilde{q} & \frac{\tilde{\beta}_1}{l}((1-\tilde{\beta})\nu-1)n - \lambda_1 c_1\\ \frac{\tilde{\beta}_1}{l}((1-\tilde{\beta})\nu-1)n - \lambda_1 c_1 & \frac{\tilde{\beta}_1{}^2\nu}{l}n + \lambda_1 c_1\end{bmatrix}$, $\quad \gamma = \frac{-1-\rho+2\tilde{\beta}+(1-\tilde{\beta})^2\nu}{l}$ and

$\vec{W} = \vec{W}(t) = \begin{bmatrix}W_1(t)\\ W_2(t)\end{bmatrix}$, where $W_1(t)$ and $W_2(t)$ are Wiener processes.

Generalizing Eq. (8.23), the stochastic point kinetic equation for m precursors become

$$\frac{d}{dt}\begin{bmatrix}\tilde{n}\\ \tilde{c}_1\\ \tilde{c}_2\\ \vdots\\ \tilde{c}_m\end{bmatrix} = \tilde{\hat{A}}\begin{bmatrix}\tilde{n}\\ \tilde{c}_1\\ \tilde{c}_2\\ \vdots\\ \tilde{c}_m\end{bmatrix} + \begin{bmatrix}\tilde{q}\\ 0\\ 0\\ \vdots\\ 0\end{bmatrix} + \tilde{\hat{B}}^{\frac{1}{2}}\frac{d\vec{W}}{dt} \tag{8.30}$$

where,

$$\tilde{\hat{A}} = \begin{bmatrix} \frac{\rho - \tilde{\beta}}{l} & \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \frac{\tilde{\beta}_1}{l} & -\lambda_1 & 0 & \cdots & 0 \\ \frac{\tilde{\beta}_2}{l} & 0 & -\lambda_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \frac{\tilde{\beta}_m}{l} & 0 & \cdots & 0 & -\lambda_m \end{bmatrix} \text{ and } \tilde{\hat{B}} = \begin{bmatrix} \tilde{\zeta} & \tilde{a}_1 & \tilde{a}_2 & \cdots & \tilde{a}_m \\ \tilde{a}_1 & \tilde{r}_1 & \tilde{b}_{2,3} & \cdots & \tilde{b}_{2,m} \\ \tilde{a}_2 & 0 & \tilde{r}_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \tilde{b}_{m-1,m} \\ \tilde{a}_m & \tilde{b}_{m,2} & \cdots & \tilde{b}_{m,m-1} & \tilde{r}_m \end{bmatrix}$$

Here, $\tilde{\zeta} = \gamma n + \sum_{i=1}^{m} \lambda_i c_i + \tilde{q}$, $\tilde{a}_i = \frac{\tilde{\beta}_i}{l}((1-\tilde{\beta})\nu - 1)n - \lambda_i c_i$, $\tilde{b}_{i,j} = \frac{\tilde{\beta}_{i-1}\tilde{\beta}_{j-1}\nu}{l} n$, and $\tilde{r}_i = \frac{\tilde{\beta}_i^{\,2}\nu}{l} n + \lambda_i c_i$.

In compact form Eq. (8.30) may be written as

$$\frac{d}{dt}\tilde{x} = \tilde{\hat{A}}\tilde{x} + [\tilde{q} \quad 0 \quad \cdots \quad 0]^T + \tilde{\hat{B}}^{\frac{1}{2}} \frac{d\vec{W}}{dt} \tag{8.31}$$

where, $\tilde{x} = [\tilde{n} \quad \tilde{c}_1 \quad \tilde{c}_2 \quad \cdots \quad \tilde{c}_m]^T$.

8.3.2 Case Study

Few authors have investigated stochastic point kinetic neutron diffusion equation by using different methods viz. Monte Carlo and Stochastic PCA etc. As mentioned previously, here we have considered the fuzziness in the parameters of the stochastic differential equation and investigated an example problem. Following are the crisp parameters used in the problem, $\lambda_1 = 0.1$, $\nu = 2.5$, $l = \frac{2}{3}$ and $\rho = -\frac{1}{3}$ for $t \geq 0$. Whereas, uncertain parameters in term of TFN are taken as $\tilde{\beta} = \tilde{\beta}_1 = [0.02, 0.05, 0.08]$, $\tilde{q} = [195, 200, 205]$ and $\tilde{x}(0) = \left[[390, 400, 410] \quad [290, 300, 310]\right]^T$.

The hybridisation of stochastic and fuzziness has been investigated in various cases depending upon the combination of uncertain parameters viz. one, two and three (all) parameters as fuzzy. Uncertain point kinetic stochastic differential equation has been studied by using the developed fuzzy Euler Maruyama method. The distribution of randomness in the neutron and neutron precursor population are depicted in Figures 8.26 to 8.53.

Firstly uncertain neutron population has been discussed following in three different cases viz. 1 to 3.

Case 1

In this case only one parameter has been considered as fuzzy and the problem is solved by using proposed hybrid fuzzy Euler Maruyama method for two samples. In Figures 8.26 and 8.27, the uncertain results of neutron population are presented where initial condition is taken as fuzzy and it is seen that the distribution of left, right and centre are different. Further, neutron source is assumed as fuzzy and the obtained uncertain neutron population has been shown in Figures 8.28 and 8.29. Finally, neutron precursor constant (β) is assumed as fuzzy and the uncertain neutron population are presented graphically in Figures 8.30 and 8.31.

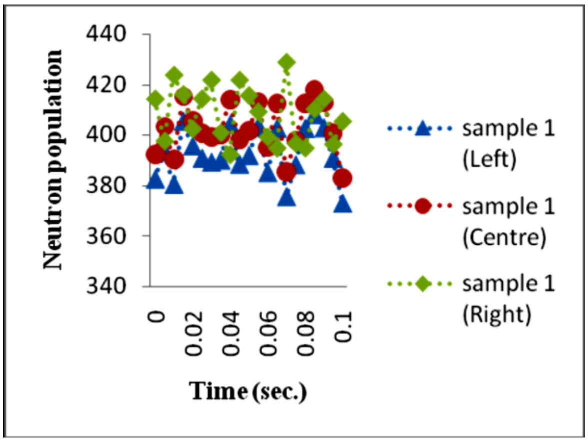

Figure 8.26. Initial condition fuzzy (sample 1)

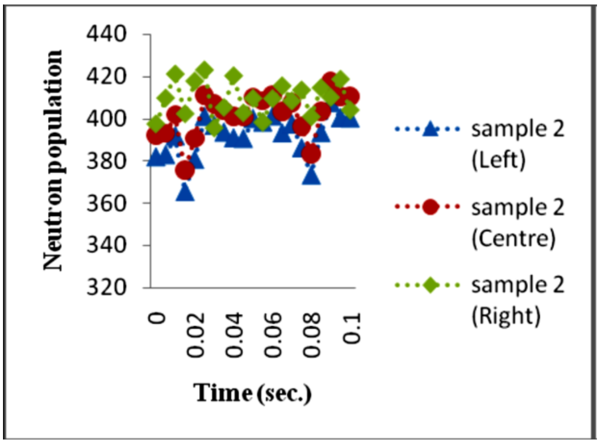

Figure 8.27. Initial condition fuzzy (sample 2)

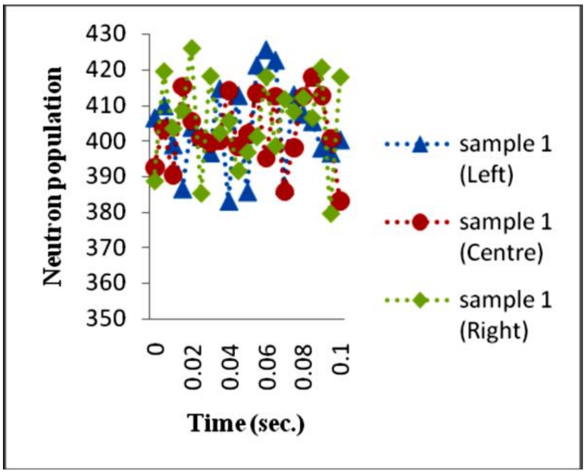

Figure 8.28. Source as fuzzy (sample 1)

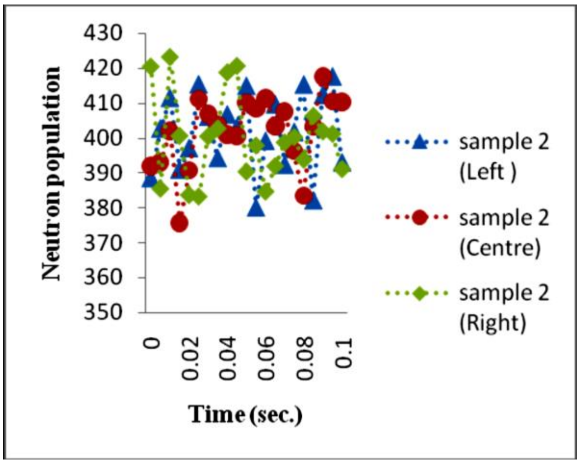

Figure 8.29. Source as fuzzy (sample 2)

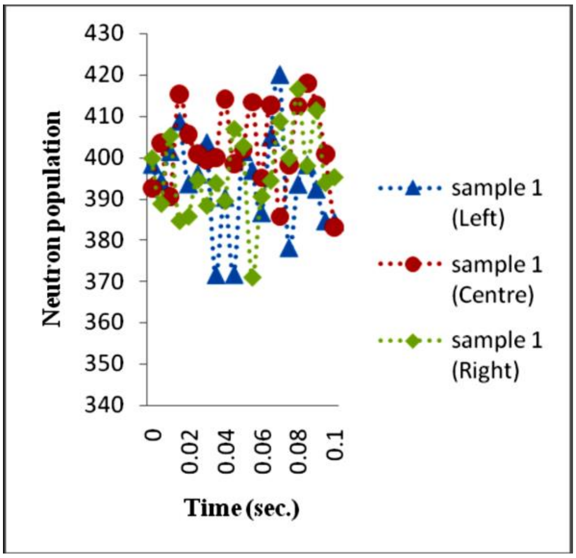

Figure 8.30. Neutron precursor constant as fuzzy (sample 1)

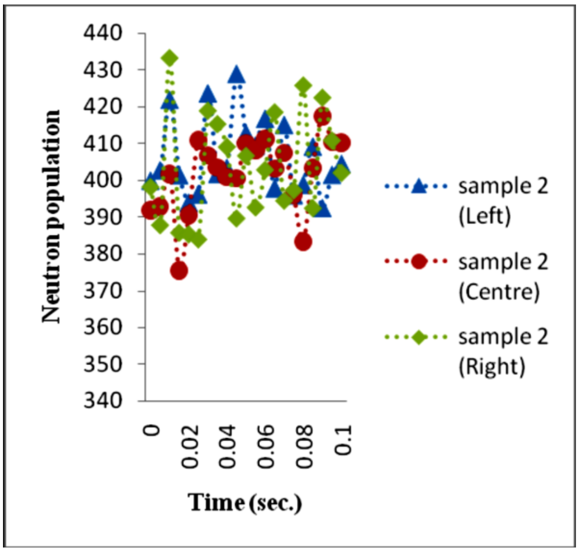

Figure 8.31. Neutron precursor constant fuzzy (sample 2)

The above graphical solutions are given in tabular form for better visualization of the uncertainty in Table 8.6. From Table 8.6, one may observe that when only initial condition is taken as fuzzy, the uncertain width is very less for both the samples in various sub cases. Hence we may conclude that when only one parameter is considered as fuzzy, uncertain initial condition is most sensitive whereas uncertain source is least sensitive.

Case 2

Here two parameters are assumed as fuzzy and various combinations of two parameters are, i) initial condition and neutron precursor constant, ii) initial condition and source, and iii) neutron precursor constant and source. In Figures 8.32 and 8.33 initial condition and neutron precursor constant are taken as fuzzy and the distribution of uncertain neutron population has been presented. Whereas in Figures 8.34 and 8.35, initial condition and source are fuzzy, and uncertain neutron populations are shown. Finally, neutron precursor constant and source are considered as fuzzy and the uncertain results are depicted in Figures 8.36 and 8.37.

Table 8.6. Comparison of neutron population when only one parameter is fuzzy

Fuzzy parameters	Samples	Expectations (Hayes & Allen 2005)	Monte Carlo (crisp) (Hayes & Allen 2005)	FEMM		Uncertain width	Stochastic PCA (crisp) (Hayes & Allen 2005)
				TFN	$\alpha = 0$		
Initial condition	Sample 1	$E(n(2))$	400.03	[392.60, 402.53, 408.18]	[392.60, 408.18]	15.58	395.32
	Sample 2	$E(n(2))$		[391.88, 401.82, 409.65]	[391.88, 409.65]	17.77	
Source	Sample 1	$E(n(2))$	400.03	[402.53, 403.65, 405.68]	[402.53, 405.68]	3.15	395.32
	Sample 2	$E(n(2))$		[399.93, 401.64, 401.82]	[399.93, 401.82]	1.89	
Neutron precursor constant	Sample 1	$E(n(2))$	400.03	[393.87, 396.13, 402.53]	[393.87, 402.53]	8.66	395.32
	Sample 2	$E(n(2))$		[401.82, 403.36, 406.04]	[401.82, 406.04]	4.22	

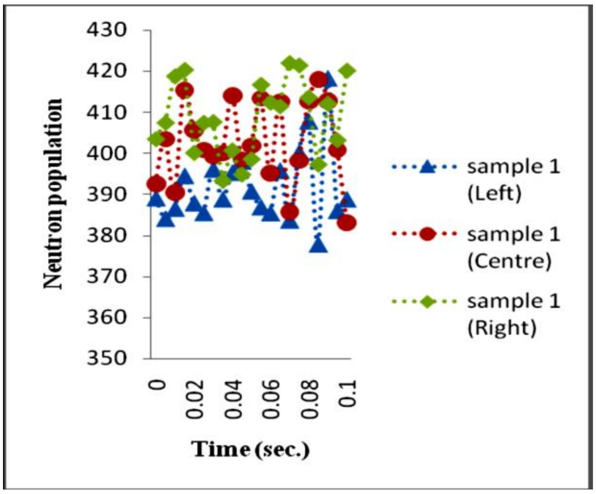

Figure 8.32. Initial condition and neutron precursor constant as fuzzy (sample 1)

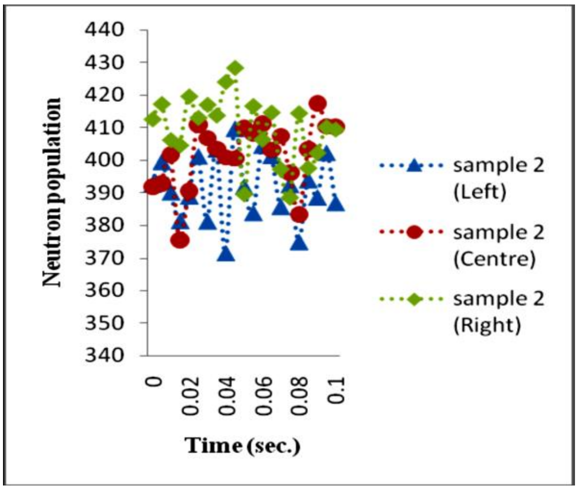

Figure 8.33. Initial condition and neutron precursor constant as fuzzy (sample 2)

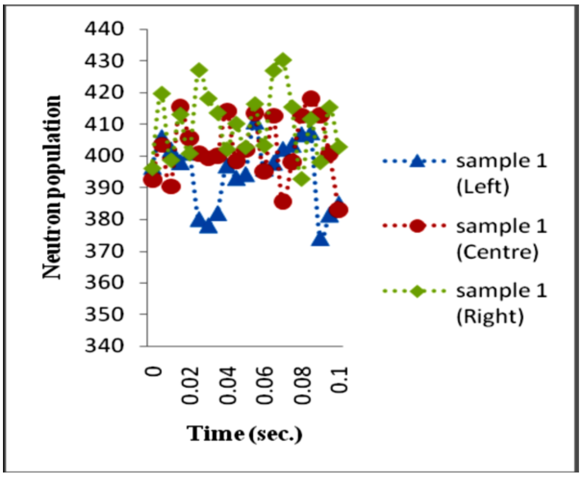

Figure 8.34. Initial condition and source as fuzzy (sample 1)

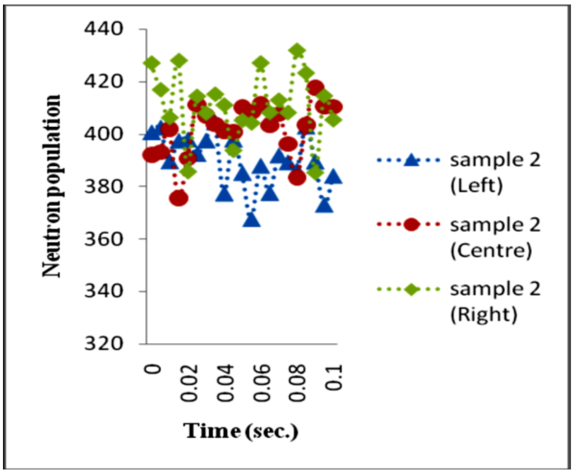

Figure 8.35. Initial condition and source as fuzzy (sample 2)

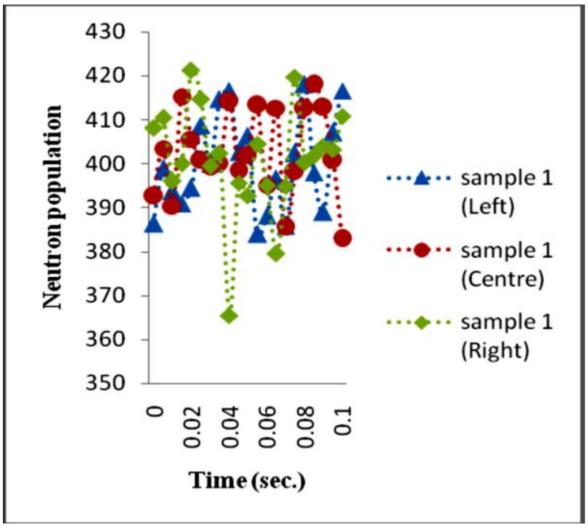

Figure 8.36. Neutron precursor constant and source as fuzzy (sample 1)

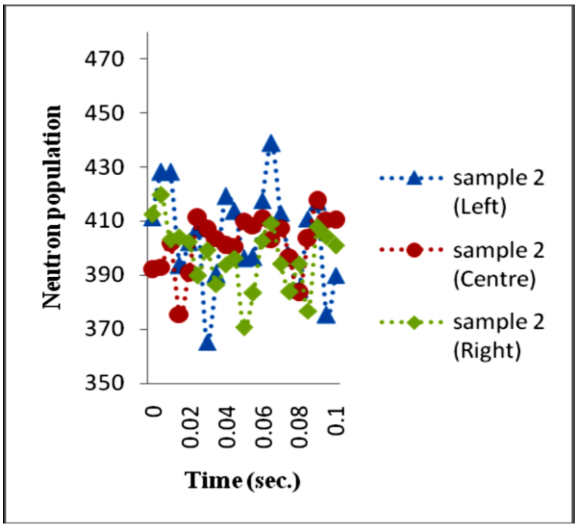

Figure 8.37. Neutron precursor constant and source as fuzzy (sample 2)

The uncertain widths of triangular fuzzy neutron population may clearly be investigated by Table 8.7. It has been seen that the combination of neutron precursor constant and source as fuzzy has less width than other sub cases for both the samples. So we may now be concluded that the combination of neutron precursor constant and source is less sensitive than other combinations.

Table 8.7. Comparison of neutron population when only two parameter are fuzzy

Fuzzy parameters	Samples	Expectations (Hayes & Allen 2005)	Monte Carlo (crisp) (Hayes & Allen 2005)	FEMM		Uncertain width	Stochastic PCA (crisp) (Hayes & Allen 2005)
				TFN	$\alpha = 0$		
Initial condition and neutron precursor constant	Sample 1	$E(n(2))$	400.03	[391.81, 402.53, 408.70]	[391.81, 408.70]	16.89	395.32
	Sample 2	$E(n(2))$		[391.76, 401.82, 409.50]	[391.76, 409.50]	17.74	
Initial condition and source	Sample 1	$E(n(2))$	400.03	[394.98, 402.53, 410.10]	[394.98, 410.10]	15.12	395.32
	Sample 2	$E(n(2))$		[390.03, 401.82, 411.05]	[390.03, 411.05]	21.02	
Source and neutron precursor constant	Sample 1	$E(n(2))$	400.03	[399.91, 400.92, 402.53]	[399.91, 402.53]	2.62	395.32
	Sample 2	$E(n(2))$		[396.90, 401.82, 404.60]	[396.90, 404.60]	7.7	

Case 3

Finally, in case 3, all the parameters viz. initial condition, source and precursor constant are taken as fuzzy. Obtained uncertain neutron population for samples 1 and 2 are shown in Figures 8.38 and 8.39 respectively. Further the uncertain width of the samples has been reported in Table 8.8.

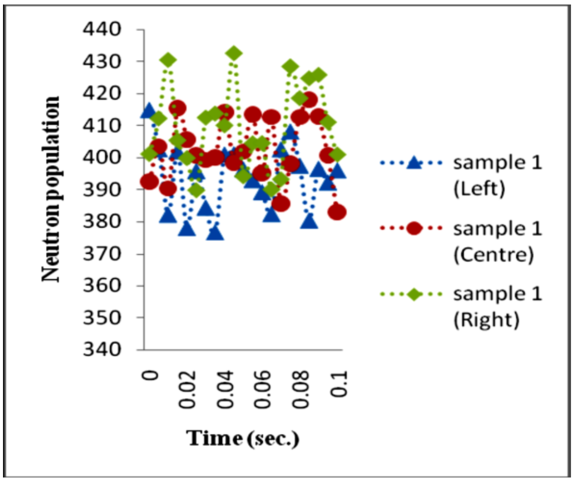

Figure 8.38. Initial condition, source and neutron precursor constant as fuzzy (sample 1)

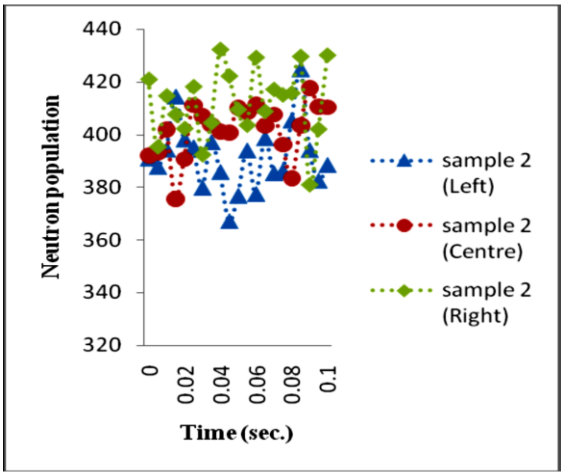

Figure 8.39. Initial condition, source and neutron precursor constant as fuzzy (sample 2)

Table 8.8. Comparison of neutron population when all parameter are fuzzy

Fuzzy parameters	Samples	Expectations (Hayes & Allen 2005)	Monte Carlo (crisp) (Hayes & Allen 2005)	FEMM		Uncertain width	Stochastic PCA (crisp) (Hayes & Allen 2005)
				TFN	$\alpha = 0$		
All fuzzy	Sample 1	$E(n(2))$	400.03	[393.93, 402.53, 409.59]	[393.93, 409.59]	15.66	395.32
	Sample 2	$E(n(2))$		[391.49, 401.82, 411.87]	[391.49, 411.87]	20.38	

Similarly, we have investigated the uncertain neutron precursor population of the system and these are discussed in cases 4 to 6 below depending upon various parameters as fuzzy.

<u>Case 4</u>

In this case, only one parameter has been considered as fuzzy and others are crisp. In Figures 8.40 and 8.41, initial condition is fuzzy. Here two samples have been investigated and it is seen that there is no overlapping between left, centre and right distribution of uncertain neutron precursor constant. The source term has been taken as fuzzy in Figures 8.42 and 8.43. Whereas, in Figures 8.44 and 8.45, neutron precursor constant is considered as fuzzy.

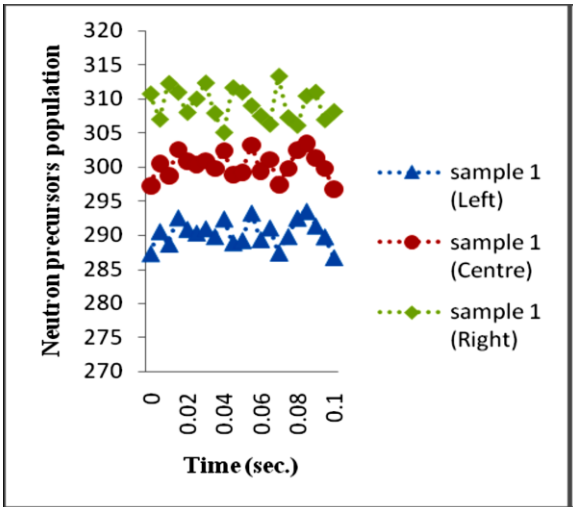

Figure 8.40. Initial condition as fuzzy (sample 1)

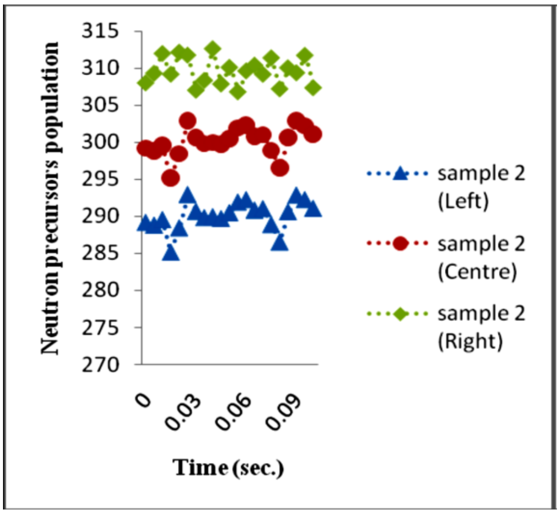

Figure 8.41. Initial condition as fuzzy (sample 2)

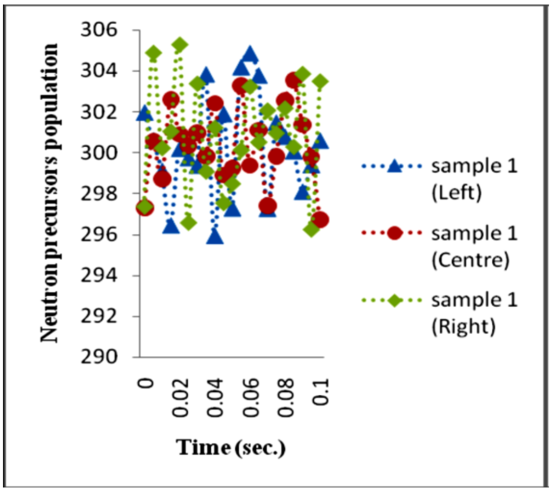

Figure 8.42. Source as fuzzy (sample 1)

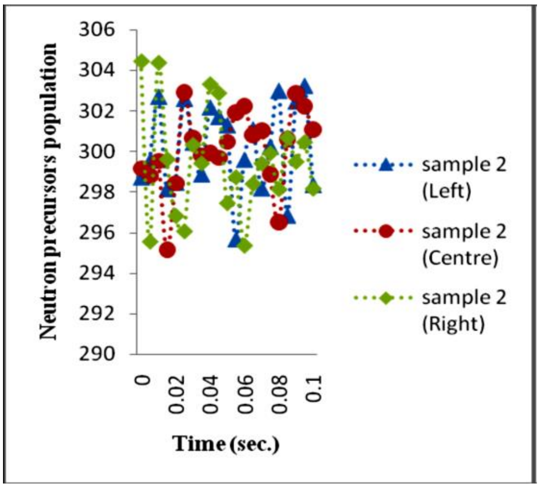

Figure 8.43. Source as fuzzy (sample 2)

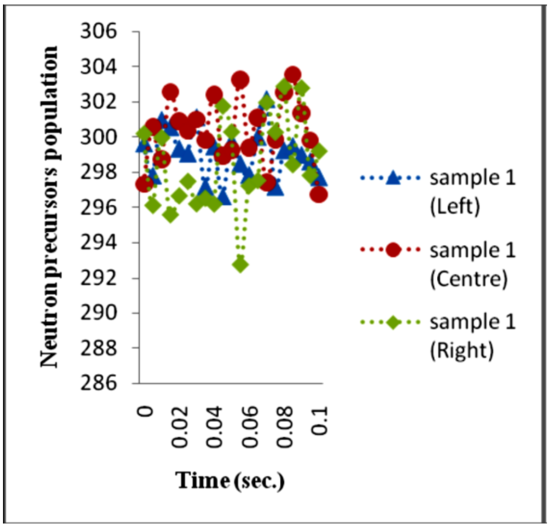

Figure 8.44. Neutron precursor constant as fuzzy (sample 1)

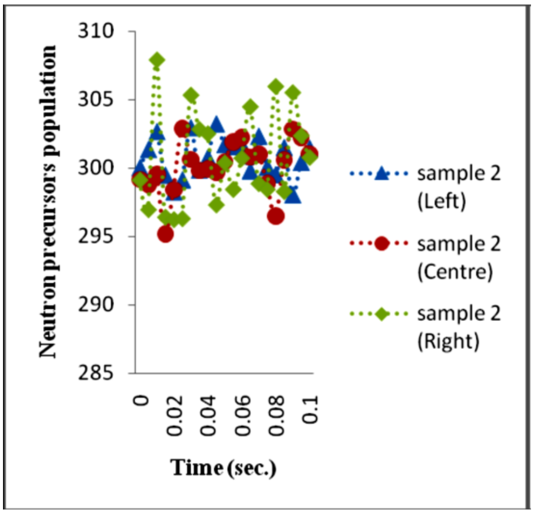

Figure 8.45. Neutron precursor constant as fuzzy (sample 2)

One may see that when initial condition is taken as fuzzy we get larger width for both the samples. Whereas, for the other problems viz. when only source and only neutron precursor constant as fuzzy then the uncertain width is less for both samples and these are presented in Table 8.9. So we may conclude that the initial condition is more sensitive.

Table 8.9. Comparison of neutron precursor population when only one parameter is fuzzy

Fuzzy parameters	Samples	Expectations (Hayes & Allen 2005)	Monte Carlo (crisp) (Hayes & Allen 2005)	FEMM		Uncertain width	Stochastic PCA (crisp) (Hayes & Allen 2005)
				TFN	$\alpha = 0$		
Initial condition	Sample 1	$E(c(2))$	300.00	[290.33, 300.32, 309.17]	[290.33, 309.17]	18.84	300.67
	Sample 2	$E(c(2))$		[290.11, 300.10, 309.62]	[290.11, 309.62]	19.51	
Source	Sample 1	$E(c(2))$	300.00	[300.32, 300.32, 300.85]	[300.32, 300.85]	0.53	300.67
	Sample 2	$E(c(2))$		[299.48, 300.10, 300.16]	[299.48, 300.16]	0.68	
Neutron precursor constant	Sample 1	$E(c(2))$	300.00	[298.44, 299.10, 300.32]	[298.44, 300.32]	1.88	300.67
	Sample 2	$E(c(2))$		[300.10, 300.69, 300.73]	[300.10, 300.73]	0.63	

<u>Case 5</u>

In this case, two parameters are taken as fuzzy and other is crisp. In Figures 8.46 and 8.47, initial condition and neutron precursor constant are fuzzy. Again two samples have been observed and seen that there is no overlapping between left, centre and right distribution of uncertain neutron precursor constant. In Figures 8.48 and 8.49, initial condition and source are considered as fuzzy. Whereas, the source and neutron precursor constant are assumed as fuzzy in Figures 8.50 and 8.51.

Uncertain width of the problems viz. (i) initial condition and neutron precursor constant are fuzzy, (ii) initial condition and source are fuzzy, and (iii) source and neutron precursor constant are fuzzy for both samples 1 and 2 are incorporated in Table 8.10. It may be observed that the uncertain width for (iii) is least. So we may conclude that the fuzziness in the combination of source and neutron precursor constant is less sensitive in comparison with the other two combinations.

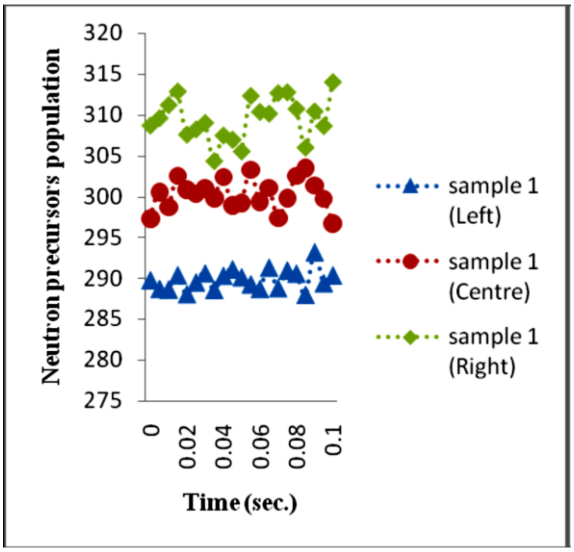

Figure 8.46. Initial condition and neutron precursor constant as fuzzy (sample 1)

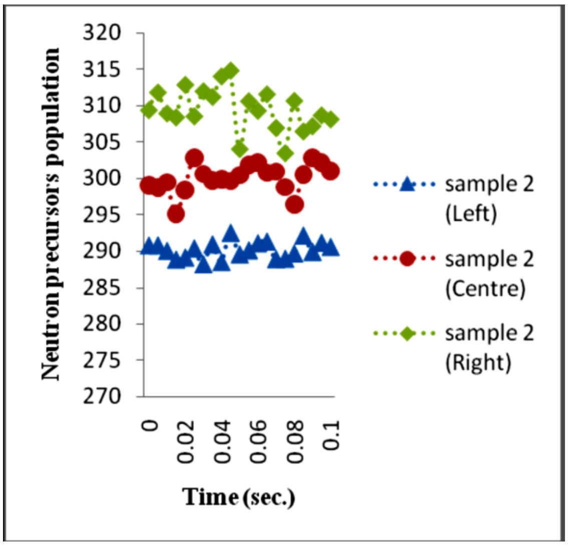

Figure 8.47. Initial condition and neutron precursor constant as fuzzy (sample 2)

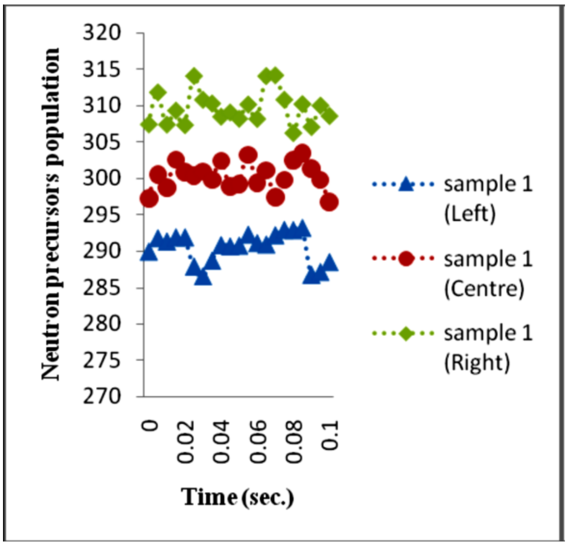

Figure 8.48. Initial condition and source as fuzzy (sample 1)

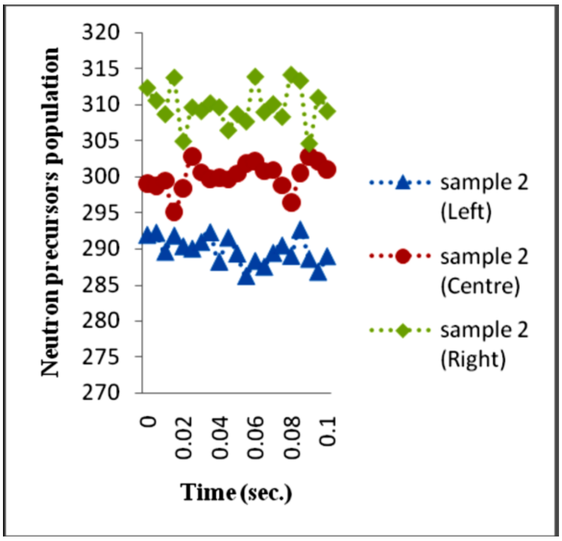

Figure 8.49. Initial condition and source as fuzzy (sample 2)

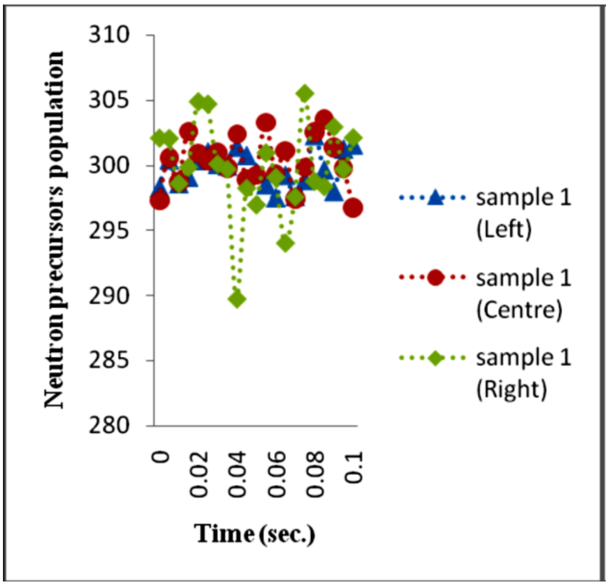

Figure 8.50. Neutron precursor constant and source as fuzzy (sample 1)

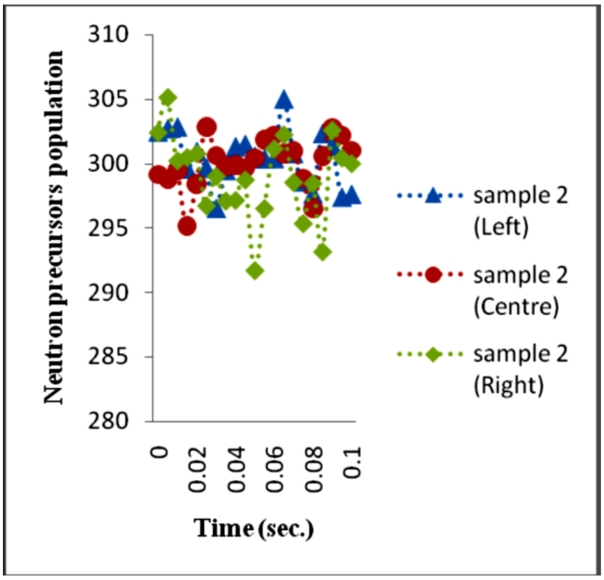

Figure 8.51. Neutron precursor constant and source as fuzzy (sample 2)

Table 8.10. Comparison of neutron precursor population when two parameter are fuzzy

Fuzzy parameters	Samples	Expectations (Hayes & Allen 2005)	Monte Carlo (crisp) (Hayes & Allen 2005)	FEMM		Uncertain width	Stochastic PCA (crisp) (Hayes & Allen 2005)
				TFN	$\alpha = 0$		
Initial condition and neutron precursor constant	Sample 1	$E(c(2))$	300.00	[289.86, 300.32, 309.50]	[289.86, 309.50]	19.64	300.67
	Sample 2	$E(c(2))$		[290.08, 300.10, 309.43]	[290.08, 309.43]	19.35	
Initial condition and source	Sample 1	$E(c(2))$	300.00	[290.43, 300.32, 309.67]	[290.43, 309.67]	19.24	300.67
	Sample 2	$E(c(2))$		[289.83, 300.10, 309.73]	[289.83, 309.73]	19.9	
Source and neutron precursor constant	Sample 1	$E(c(2))$	300.00	[299.68, 299.76, 300.32]	[299.68, 300.32]	0.49	300.67
	Sample 2	$E(c(2))$		[298.91, 300.10, 300.38]	[298.91, 300.38]	1.47	

Case 6

Here all three parameters are considered as fuzzy, and left, right and centre distribution of uncertain neutron precursor population have been shown in Figures 8.52 and 8.53. Further, in Table 8.11, uncertain widths of the neutron precursor population for the samples are presented.

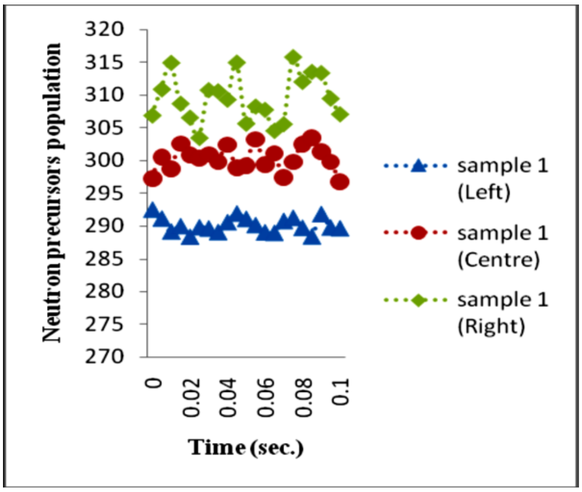

Figure 8.52. Initial condition, source and neutron precursor constant as fuzzy (sample 1)

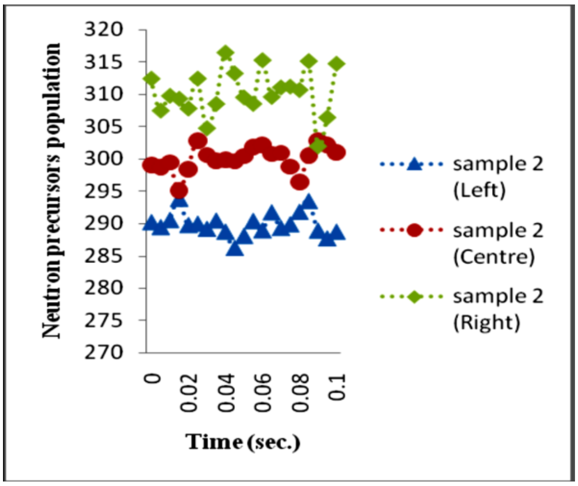

Figure 8.53. Initial condition, source and neutron precursor constant as fuzzy (sample 2)

Table 8.11. Comparison of neutron precursor population when all parameters are fuzzy

Fuzzy parameters	Samples	Expectations (Hayes & Allen 2005)	Monte Carlo (crisp) (Hayes & Allen 2005)	FEMM		Uncertain width	Stochastic PCA (crisp) (Hayes & Allen 2005)
				TFN	$\alpha = 0$		
All fuzzy	Sample 1	$E(c(2))$	300.00	[290.16, 300.32, 309.51]	[290.16, 309.51]	19.35	300.67
	Sample 2	$E(c(2))$		[289.90, 300.10, 310.22]	[289.90, 310.22]	20.32	

The main aim of this work is to demonstrate the proposed method and investigate fuzzy stochastic neutron diffusion equation. Sensitivity of the solution is observed by considering one, two and three parameters as fuzzy. The variations of uncertain widths are investigated for two random samples and results are reported in both tabular form and graphically for better visualization. Further it has been studied that the uncertainty of the solution varies problem to problem and case to case. So one may choose or taken care of the fuzzy parameters accordingly.

8.4 Conclusion

In this chapter the concept of stochasticity and fuzziness are combined to model the discussed problems and then numerical methods are used to solve the same. Limit method is applied as a tool in the computation for solving the fuzzy stochastic differential equations. The proposed methods viz. FEMM and FMM are demonstrated by some standard problems. The solutions of uncertain width along with the left and right bounds of the fuzzy stochastic differential equation have been investigated for various cases. Further, fuzzy stochastic model has been analysed by considering fuzzy stochastic point kinetic equations. Different combinations of parameters are considered as fuzzy along with the randomness of the system. Proposed Fuzzy Euler Maruyama Method (FEMM) has been used to investigate the fuzzy stochastic point kinetic equation. Depending upon various combinations of fuzzy parameters, the uncertain neutron density and delayed neutron population are obtained with respect to different cases. The uncertain neutron density and delayed neutron population are compared with Monte Carlo and stochastic PCA methods. Sensitivity of the fuzzy parameters with stochastic behaviour has also been analysed for each case.

Chapter 9

Fuzzy Wavelet Theory for Solving Differential Equations

The contents of this chapter have been communicated/accepted in the following journal/book:

1. S. Nayak and S. Chakraverty, Fuzzy wavelet method for solving ordinary differential equation with uncertain parameters, ***Mathematics and Computers in Simulation***;
2. S. Nayak and S. Chakraverty, Interval Wavelet Method in Solving Diffusion Equations, Generalized and Hybrid Set Structures and Applications for Soft Computing, ***IGI Publication***, USA, edited by Sunil Jacob John, 2015.

Chapter 9

Fuzzy Wavelet Theory for Solving Differential Equations

Recently, Wavelet Method (WM) is becoming a powerful tool to solve various types of differential equations. Till date WM has been used for crisp problems that is where the variables and parameters in the differential equations are considered as crisp. But in general the real world problems, every system contains uncertainty and this makes the corresponding mathematical model uncertain. The uncertain and imprecise parameters make the system complex.

Here the concept of interval/fuzzy theory has been combined with wavelet method and Interval/Fuzzy Wavelet Method (I/FWM) has been developed for the first time. Using I/FWM a simple pedagogic diffusion equation has been studied. Obtained results are compared with the crisp wavelet and exact solution. It has also been seen that if the resolution level of the wavelet is increased then we get better approximation. Due to the simplicity of this method, it may be used for various other diffusion problems.

9.1 Interval Wavelet Method (IWM)

Let us consider Haar wavelet family as (Lepik, 2007)

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [s_1, s_2) \\ -1 & \text{for } x \in [s_2, s_3) \\ 0 & \text{elsewhere} \end{cases} \tag{9.1}$$

where $s_1 = \dfrac{k}{m}, s_2 = \dfrac{k+0.5}{m}, s_3 = \dfrac{k+1}{m}$, integer $m = 2^j$, $j = 0, 1, ..., J$ is the resolution level of the wavelet; $k = 0, 1, ..., m-1$ is the translation parameter; the minimal of i is 2, $i = m + k + 1$ and maximal value is $2M = 2^{J+1}$. Then the scaling function $\phi(x) = h_1(x)$ is defined as

$$\phi(x) = h_1(x) = \begin{cases} 1 & \text{for } x \in [0, 1] \\ 0 & \text{elsewhere} \end{cases} \tag{9.2}$$

For $i = 2$, we get the following mother wavelet

$$\psi(x) = h_2(x) = \begin{cases} 1 & \text{for } x \in [0, 1/2) \\ -1 & \text{for } x \in [1/2, 1) \\ 0 & \text{elsewhere} \end{cases} \tag{9.3}$$

The scaling function and its translate are shown in Figure 9.1. Let us assume an arbitrary function $f(x)$, which may be expressed as a linear combination of translations; $f(x) = 2\phi(x) + 3\phi(x-1) - 2\phi(x-2) + 4\phi(x-3)$ which has been shown in Figure 9.2.

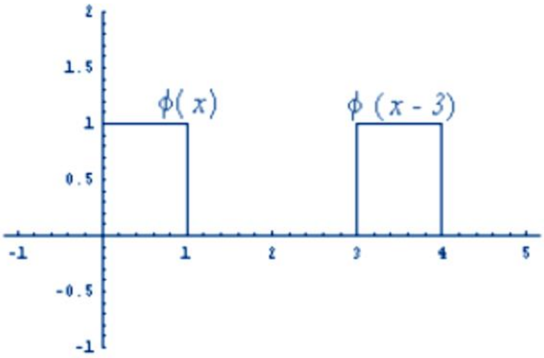

Figure 9.1. Scaling function $\phi(x)$ and its translation

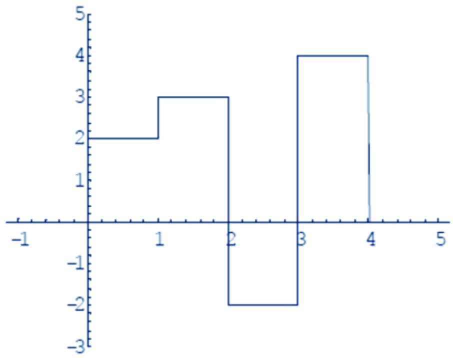

Figure 9.2. A linear combination of translations

The shrinkage of the scaling function is represented in Figure 9.3 which is further used to approximate the function $f(x)$ and is shown in Figure 9.4.

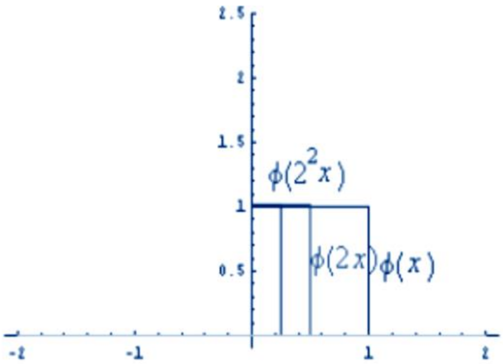

Figure 9.3. Scaling function $\phi(x)$ and its shrinkage

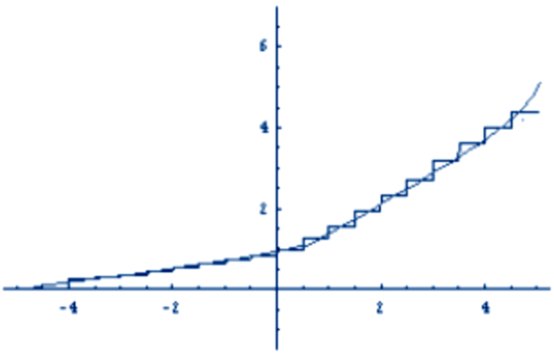

Figure 9.4. Approximation of a function $f(x)$ by translate $\phi(2x)$

Let us consider $V_0 =$ all square integrable linear combinations of function ϕ and its translates

$$= \left\{ f(x) = \sum_k a_k \phi_{0k}(x) \middle| a_k \in R;\ f \in L^2 \right\}$$

and $V_1 =$ all square integrable linear combinations of function ϕ and its translates

$$= \left\{ f(x) = \sum_k a_k \phi_{1k}(x) \middle| a_k \in R;\ f \in L^2 \right\}.$$

Then we may write $V_0 \subset V_1$ and

$$\begin{aligned} \phi(x) &= \phi(2x) + \phi(2x-1) \\ &= \frac{1}{\sqrt{2}}\phi_{10}(x) + \frac{1}{\sqrt{2}}\phi_{11}(x) \\ &= a_{10}\phi_{10}(x) + a_{11}\phi_{11}(x) \end{aligned} \tag{9.4}$$

where $a_{10} = \frac{1}{\sqrt{2}}$; $a_{11} = \frac{1}{\sqrt{2}}$ and all other a's are zero.

The expression (9.4) has been depicted in Figure 9.5.

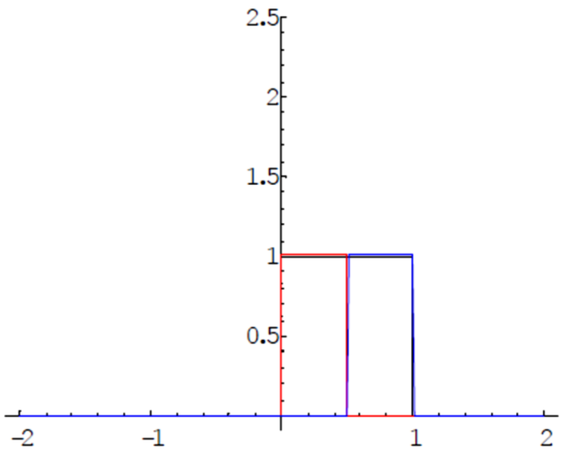

Figure 9.5. A linear combination $\phi(x) = \phi(2x) + \phi(2x-1)$

Generalizing the concept of Haar wavelet functions, we may write

$$p_i(x) = \int_0^x h_i(x)dx = \begin{cases} x - s_1 & \text{for } x \in [s_1, s_2) \\ s_3 - x & \text{for } x \in [s_2, s_3) \\ 0 & \text{elsewhere} \end{cases} \tag{9.5}$$

and

$$q_i(x) = \int_0^x p_i(x)dx = \begin{cases} 0 & \text{for } x \in [0, s_1) \\ \frac{1}{2}(x - s_1)^2 & \text{for } x \in [s_1, s_2) \\ \frac{1}{4m^2} - \frac{1}{2}(s_3 - x)^2 & \text{for } x \in [s_2, s_3) \\ \frac{1}{4m^2} & \text{for } x \in [s_3, 1] \end{cases} \tag{9.6}$$

Let us divide the interval $x \in [0,1]$ into $2M$ parts of equal length $\Delta x = \frac{1}{2M}$. Then the grid and collocation points are $\zeta_l = (l-1)\Delta x$ and $x_l = (l-0.5)\Delta x, \quad l = 1, 2, ..., 2M$ respectively. Using these collocation points we get $2M \times 2M$ coefficient matrices $H(i,l) = h_i(x_l)$, $P(i,l) = p_i(x_l)$ and $Q(i,l) = q_i(x_l)$.

The interval values are then computed in place of crisp parameters and the crisp expressions are investigated in terms of interval. Due to involvement of interval uncertainty we get uncertain differential equations and uncertain matrices H, P, Q in terms of interval. These interval matrices may be written as follows

$$\begin{aligned} H^I(i,l) &= [\underline{H}(i,l), \overline{H}(i,l)] \\ &= [\underline{h_i}, \overline{h_i}] \end{aligned} \tag{9.7}$$

$$\begin{aligned} P^I(i,l) &= [\underline{P}(i,l), \overline{P}(i,l)] \\ &= [\underline{p_i}, \overline{p_i}] \end{aligned} \tag{9.8}$$

$$\begin{aligned} Q^I(i,l) &= [\underline{Q}(i,l), \overline{Q}(i,l)] \\ &= [\underline{q_i}, \overline{q_i}] \end{aligned} \tag{9.9}$$

9.2 Case Study

Let us consider a simple differential equation

$$\frac{d^2u}{dx^2} - u = 0; \quad u(0) = 0.5, u'(0) = 2.5 \tag{9.10}$$

The exact solution of Eq. (9.10) is

$$u = c_1 e^x + c_2 e^{-x} \tag{9.11}$$

Now using the initial condition $u(0) = 0.5, u'(0) = 2.5$ the exact solution as $u = \frac{3}{2}e^x - e^{-x}$.

9.2.1 Crisp Haar Wavelet

Let us fix the value of $J = 1$. Then depending on the value of J, we get $m = 1, 2$ and $k = 0, 1$. For different values of k, m, J and i following matrices may easily be obtained.

$$h_1(x) = \begin{cases} 1 & \text{for } x \in [0,1] \\ 0 & \text{elsewhere} \end{cases} \qquad h_2(x) = \begin{cases} 1 & \text{for } x \in [0, 1/2) \\ -1 & \text{for } x \in [1/2, 1) \\ 0 & \text{elsewhere} \end{cases}$$

$$h_3(x)=\begin{cases}1 & \text{for } x\in[0,1/4)\\ -1 & \text{for } x\in[1/4,1/2)\\ 0 & \text{elsewhere}\end{cases}\qquad h_4(x)=\begin{cases}1 & \text{for } x\in[1/2,3/4)\\ -1 & \text{for } x\in[3/4,1)\\ 0 & \text{elsewhere}\end{cases}$$

$$p_1(x)=\begin{cases}x & \text{for } x\in[0,1)\\ 0 & \text{elsewhere}\end{cases}\qquad p_2(x)=\begin{cases}x & \text{for } x\in[0,1/2)\\ 1-x & \text{for } x\in[1/2,1)\\ 0 & \text{elsewhere}\end{cases}$$

$$p_3(x)=\begin{cases}x & \text{for } x\in[0,1/4)\\ \frac{1}{2}-x & \text{for } x\in[1/4,1/2)\\ 0 & \text{elsewhere}\end{cases}\qquad p_4(x)=\begin{cases}x-\frac{1}{2} & \text{for } x\in[1/2,3/4)\\ 1-x & \text{for } x\in[3/4,1)\\ 0 & \text{elsewhere}\end{cases}$$

$$q_1(x)=\begin{cases}\frac{x^2}{2} & \text{for } x\in[0,1)\\ 0 & \text{elsewhere}\end{cases}\qquad q_2(x)=\begin{cases}0 & \text{for } x\in[0,0)\\ \frac{1}{2}x^2 & \text{for } x\in[0,1/2)\\ \frac{1}{4}-\frac{1}{2}(1-x)^2 & \text{for } x\in[1/2,1)\\ \frac{1}{4} & \text{for } x\in[1,1]\end{cases}$$

$$q_3(x)=\begin{cases}0 & \text{for } x\in[0,0)\\ \frac{1}{2}x^2 & \text{for } x\in[0,1/4)\\ \frac{1}{16}-\frac{1}{2}\left(\frac{1}{4}-x\right)^2 & \text{for } x\in[1/4,1/2)\\ \frac{1}{16} & \text{for } x\in[1/2,1]\end{cases}\qquad q_4(x)=\begin{cases}0 & \text{for } x\in[0,1/2)\\ \frac{1}{2}\left(x-\frac{1}{2}\right)^2 & \text{for } x\in[1/2,3/4)\\ \frac{1}{16}-\frac{1}{2}(1-x)^2 & \text{for } x\in[3/4,1)\\ \frac{1}{16} & \text{for } x\in[1,1]\end{cases}$$

Using collocation points $x_l=(l-0.5)\Delta x,\quad l=1,2,3,4$, matrices H,P,Q have been evaluated and are represented as below.

$$H=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & -1\end{bmatrix},\ P=\frac{1}{8}\begin{bmatrix}1 & 3 & 5 & 7\\ 1 & 3 & 3 & 1\\ 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix} \text{and } Q=\frac{1}{128}\begin{bmatrix}1 & 9 & 25 & 49\\ 1 & 9 & 23 & 31\\ 1 & 7 & 8 & 8\\ 0 & 0 & 1 & 7\end{bmatrix}.$$

It has been investigated in the following way.

Consider a linear combination of Haar wavelet function as the highest order derivative and then integrate it.

$$\frac{d^2u}{dx^2} = \sum_i a_i h_i(x)$$

$$\Rightarrow u'(x) - u(0) = \sum_i a_i p_i(x)$$

$$\Rightarrow u(x) = \sum_i a_i q_i(x) + xu'(0) + u(0) \tag{9.12}$$

Further Eq. (9.12) is substituted in Eq. (9.10), and we obtain

$$\sum_i a_i h_i(x) - \sum_i a_i q_i(x) - xu'(0) + u(0) = 0$$

Putting collocation points x_l on the above equation we obtain the values of a_i and then one may investigate the required solution.

9.2.2 Interval Haar Wavelet

Due to the presence of interval parameters the following interval differential equation is obtained. Here, only the system coefficient has been taken as interval and other parameters as crisp.

$$\frac{d^2u}{dx^2} - [\underline{b}, \overline{b}]u = 0\ ; \qquad [\underline{b}, \overline{b}] = [0.5, 1.5],\ u(0) = 0.5,\ u'(0) = 2.5 \tag{9.13}$$

First consider the left value of the interval parameter and Eq. (9.13) may be represented as

$$\frac{d^2u}{dx^2} - \underline{b}u = 0$$

$$\Rightarrow \sum_i a_i h_i(x) - \underline{b}\sum_i a_i q_i(x) - x\underline{b}u'(0) + \underline{b}u(0) = 0$$

$$\Rightarrow \{A\}[H] - \underline{b}\{A\}[Q] - \{X\}\underline{b}u'(0) + \underline{b}u(0) = 0 \tag{9.14}$$

where $\{A\} = \{a_i\}, \{X\} = \{x_l\}, [H] = [h_i(x_l)]$ and $[Q] = [q_i(x_l)]$

Now putting the collocation points x_l on the Eq. (9.14), the left solution will be

$$\underline{b}\{A\}[Q(x)] - x\underline{b}u'(0) + \underline{b}u(0) = 0 \tag{9.15}$$

Next considering the right value of the interval parameter, Eq. (9.13) may be written as

$$\frac{d^2u}{dx^2} - \overline{b}u = 0$$

$$\Rightarrow \sum_i a_i h_i(x) - \overline{b}\sum_i a_i q_i(x) - x\overline{b}u'(0) + \overline{b}u(0) = 0$$

$$\Rightarrow \{A\}[H] - \overline{b}\{A\}[Q] - \{X\}\overline{b}u'(0) + \overline{b}u(0) = 0 \tag{9.16}$$

Now putting the collocation points x_l on Eq. (9.16) we get the right solution

$$\bar{b}\{A\}[Q(x)] - x\bar{b}u'(0) + \bar{b}u(0) = 0 \tag{9.17}$$

Combining both left and right solutions the general solution in terms of matrix form may be written as

$$\{A\}[Q^I(x)] - [\underline{b}, \bar{b}]xu'(0) + [\underline{b}, \bar{b}]u(0) = 0 \tag{9.18}$$

For various values of *x*, different set of interval solutions are investigated. A comparison of exact, crisp wavelet and interval wavelet solution is given in Table 9.1.

Table 9.1. Comparison of exact, crisp wavelet and interval wavelet solution

	Exact solution	Crisp wavelet solution	Interval wavelet solution
$u\left(\frac{1}{8}\right)$	0.8172	0.8110	[0.4055, 1.2165]
$u\left(\frac{3}{8}\right)$	1.4951	1.4290	[0.7146, 2.1434]
$u\left(\frac{5}{8}\right)$	2.2671	2.0717	[1.0372, 3.1107]
$u\left(\frac{7}{8}\right)$	3.1814	2.7704	[1.3933, 4.1803]

From the obtained solution of interval wavelet method we have seen that the mid value of the solution set is nothing but the crisp wavelet solution. Here a point to be noted that if the resolution level is increased then the solutions obtained by using crisp as well as interval wavelet method converges to exact solution.

9.2.3 Fuzzy Haar Wavelet

When fuzzy parameters are introduced in Eq. (9.13), we may have the following fuzzy differential equation.

$$\begin{gathered} \frac{d^2u}{dx^2} - \tilde{b}u = 0; \\ \tilde{b} = [0.5, 1, 1.5], u(0) = 0.5, u'(0) = 2.5. \end{gathered} \tag{9.19}$$

Here, only the system coefficient has been taken as fuzzy and other parameters are kept as crisp. Using (Nayak & Chakraverty 2013), $\tilde{b}$ may be written as $[\lim_{n\to\infty} s, \lim_{n\to 1} s]$, where

$s = \underline{b}(\alpha) + \dfrac{\bar{b}(\alpha) - \underline{b}(\alpha)}{n}$ and $\underline{b}(\alpha)$ and $\bar{b}(\alpha)$ are defined in Chapter 3. Considering the left continuous function (i.e. $\underline{b}(\alpha)$) Eq. (9.19) may be represented as

$$\frac{d^2u}{dx^2} - \lim_{n\to\infty} su = 0$$

$$\Rightarrow \sum_i a_i h_i(x) - (\lim_{n\to\infty} s)\sum_i a_i q_i(x) - x(\lim_{n\to\infty} s)u'(0) + (\lim_{n\to\infty} s)u(0) = 0$$

$$\Rightarrow \{A\}[H] - \lim_{n\to\infty} s\{A\}[Q] - \{X\}\lim_{n\to\infty} su'(0) + \lim_{n\to\infty} su(0) = 0 \tag{9.20}$$

where $\{A\} = \{a_i\}$, $\{X\} = \{x_l\}$, $[H] = [h_i(x_l)]$ and $[Q] = [q_i(x_l)]$

Next, considering the right continuous function (i.e. $\bar{b}(\alpha)$), Eq. (9.19) may be written as

$$\frac{d^2u}{dx^2} - \lim_{n\to 1} su = 0$$

$$\Rightarrow \sum_i a_i h_i(x) - (\lim_{n\to 1} s)\sum_i a_i q_i(x) - x(\lim_{n\to 1} s)u'(0) + (\lim_{n\to 1} s)u(0) = 0$$

$$\Rightarrow \{A\}[H] - \lim_{n\to 1} s\{A\}[Q] - \{X\}\lim_{n\to 1} su'(0) + \lim_{n\to 1} su(0) = 0 \tag{9.21}$$

Now putting the collocation points x_l on Eqs. (9.20) and (9.21) we get

$$\lim_{n\to\infty} s\{A\}[Q(x)] - x\lim_{n\to\infty} su'(0) + \lim_{n\to\infty} su(0) = 0 \tag{9.22}$$

$$\lim_{n\to 1} s\{A\}[Q(x)] - x\lim_{n\to 1} su'(0) + \lim_{n\to 1} su(0) = 0 \tag{9.23}$$

Denoting

$$S_1 = \lim_{n\to\infty} s\{A\}[Q(x)] - x\lim_{n\to\infty} su'(0) + \lim_{n\to\infty} su(0)$$

$$S_2 = \lim_{n\to 1} s\{A\}[Q(x)] - x\lim_{n\to 1} su'(0) + \lim_{n\to 1} su(0).$$

We have the following final uncertain solution

$$\bigcup_{\alpha}[\min(S_1, S_2), \max(S_1, S_2)] \tag{9.24}$$

For various values of *x*, different set of interval solutions are investigated. A comparison of exact and fuzzy wavelet solution is given in Table 9.2.

Table 9.2. Comparison of exact and fuzzy wavelet solution

	Exact solution	Fuzzy wavelet solution
$u\left(\frac{1}{8}\right)$	0.8172	[0.4055, 0.8151, 1.2165]
$u\left(\frac{3}{8}\right)$	1.4951	[0.7146, 1.4643, 2.1434]
$u\left(\frac{5}{8}\right)$	2.2671	[1.0372, 2.0817, 3.1107]
$u\left(\frac{7}{8}\right)$	3.1814	[1.3933, 2.7092, 4.1803]

It may be noted that the approximation of the obtained results become better as we increase the resolution level. Accordingly, the uncertain width of the solution may vary with the resolution of wavelets. Also there is a convergence of uncertain width of the solutions for particular x, when different resolutions are considered. Further, for better visualization of uncertain distribution of solutions, at different α-cut and collocation points, left and right bound of the uncertain solutions are given in Tables 9.3 to 9.6.

Table 9.3. Solutions for various values of α at $x=\frac{1}{8}$

α	Left	Right
$\alpha=0$	0.8151	0.8167
$\alpha=0.1$	0.8153	0.8168
$\alpha=0.2$	0.8155	0.8168
$\alpha=0.3$	0.8157	0.8168
$\alpha=0.4$	0.8159	0.8168
$\alpha=0.5$	0.8160	0.8168
$\alpha=0.6$	0.8161	0.8168
$\alpha=0.7$	0.8163	0.8168
$\alpha=0.8$	0.8164	0.8167
$\alpha=0.9$	0.8165	0.8167
$\alpha=1$	0.8166	0.8166

Table 9.4. Solutions for various values of α at $x=\frac{3}{8}$

α	Left	Right
$\alpha=0$	1.4560	1.4613
$\alpha=0.1$	1.4573	1.4622
$\alpha=0.2$	1.4585	1.4629
$\alpha=0.3$	1.4596	1.4635
$\alpha=0.4$	1.4606	1.4640
$\alpha=0.5$	1.4615	1.4643
$\alpha=0.6$	1.4623	1.4646
$\alpha=0.7$	1.4630	1.4647
$\alpha=0.8$	1.4635	1.4647
$\alpha=0.9$	1.4640	1.4646
$\alpha=1$	1.4643	1.4643

Table 9.5. Solutions for various values of α at $x=\frac{5}{8}$

α	Left	Right
$\alpha=0$	2.0507	2.0845
$\alpha=0.1$	2.0551	2.0854
$\alpha=0.2$	2.0593	2.0860
$\alpha=0.3$	2.0631	2.0864
$\alpha=0.4$	2.0667	2.0865
$\alpha=0.5$	2.0699	2.0864
$\alpha=0.6$	2.0728	2.0860
$\alpha=0.7$	2.0755	2.0853
$\alpha=0.8$	2.0778	2.0844
$\alpha=0.9$	2.0799	2.0832
$\alpha=1$	2.0817	2.0817

Table 9.6. Solutions for various values of α at $x=\frac{7}{8}$

α	Left	Right
$\alpha=0$	2.6511	2.7194
$\alpha=0.1$	2.6592	2.7203
$\alpha=0.2$	2.6668	2.7209
$\alpha=0.3$	2.6739	2.7210
$\alpha=0.4$	2.6805	2.7207
$\alpha=0.5$	2.6865	2.7199
$\alpha=0.6$	2.6920	2.7187
$\alpha=0.7$	2.6971	2.7170
$\alpha=0.8$	2.7016	2.7149
$\alpha=0.9$	2.7056	2.7123
$\alpha=1$	2.7092	2.7092

Tabulated values are depicted graphically in Figures 9.6 to 9.9. It may be observed that the distribution of uncertainty varies if we consider different values of α-cut. As a result, we may not get TFN solutions always. But for different values of α-cut from 0 to 1, the width of the obtained interval solutions becomes thinner and finally crisp for $\alpha = 1$.

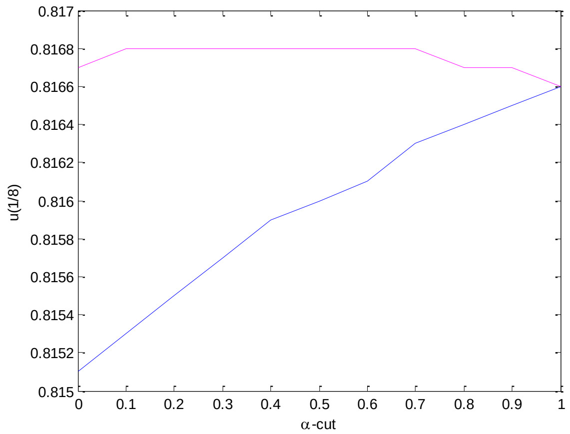

Figure 9.6. Fuzzy solution at x=1/8 (Table 9.3)

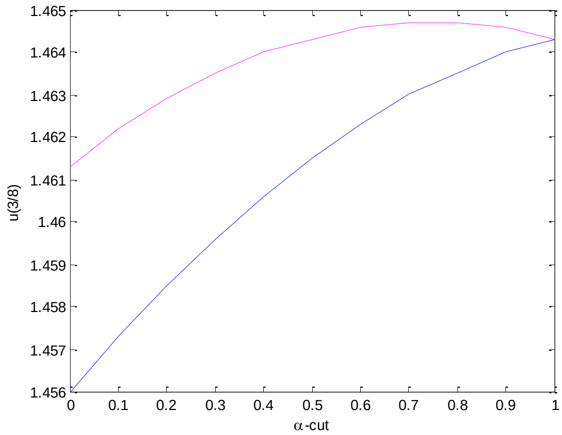

Figure 9.7. Fuzzy solution at x=3/8 (Table 9.4)

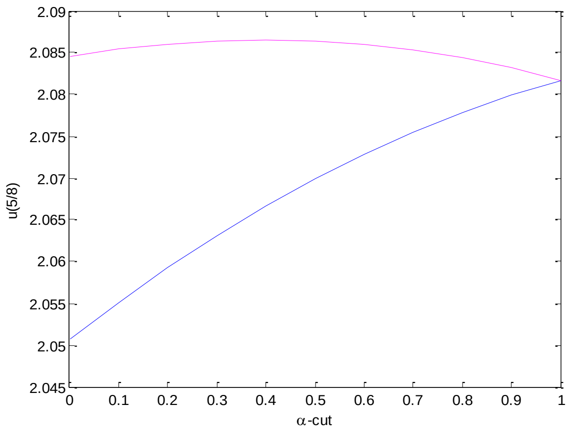

Figure 9.8. Fuzzy solution at x=5/8 (Table 9.5)

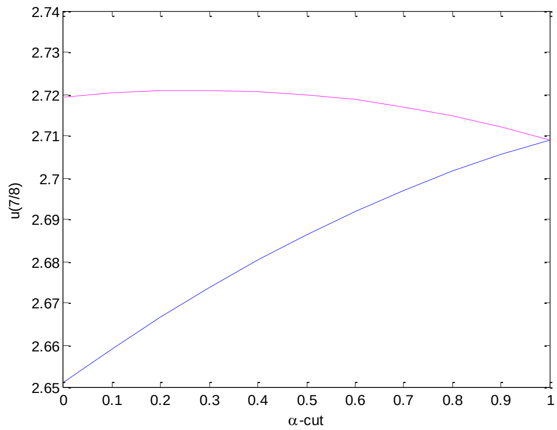

Figure 9.9. Fuzzy solution at x=7/8 (Table 9.6)

9.3 Conclusion

In this chapter a novel idea viz. Fuzzy Wavelet Method (FWM) has been proposed. Here the uncertain parameters are considered as fuzzy and then fuzzy theory has been combined with the wavelet method. An example problem has been investigated by using proposed FWM. Obtained solutions are compared with the crisp wavelet and exact solutions. Further, it may also be pointed out that we get better approximation if the resolution level of the wavelet is increased.

Chapter 10

Conclusions and Future Directions

Chapter 10

Conclusions and Future Directions

This chapter includes overall conclusions of the present study and suggestions for future work. Here, new and alternate form of interval/fuzzy arithmetic and corresponding interval/fuzzy numerical methods have been developed to handle various diffusion problems viz. heat transfer and neutron diffusion etc. with uncertainties.

In the following paragraphs, conclusions are drawn with respect to various proposed methods and application problems that have been undertaken.

10.1 Conclusions

- In this thesis, traditional interval arithmetic has been redefined in a unique way. In this procedure the interval arithmetic is modified using crisp representation of intervals. This interval arithmetic is then extended for triangular and trapezoidal fuzzy numbers using α -cut techniques. The number of computations and the time taken has been found to be less in comparison with the well known vertex method.

- The proposed interval/fuzzy arithmetic (of chapter 3) has been used to handle the uncertain parameters in the FEM and then Interval/Fuzzy Finite Element Method (I/FEFM) has been developed. A couple unsteady state heat conduction problem is solved by using FFEM in Chapter 4. Accordingly iterative and eigenvalue methods have been used to manage the difficulty occurred in the coupled differential equations. Uncertain distribution of temperatures have been studied at different level of relative errors. Further, in chapter 5, uncertain conjugate heat transfer problems have been investigated by using the proposed FFEM. The sensitiveness of the uncertain parameters have also been studied here.

- Next, the concept of interval and fuzziness have been combined with FEM to model one and multi group neutron diffusion problems in chapters 6 and 7 respectively. Accordingly a bench mark two group neutron diffusion problem has been investigated. In this study one may see that the speed of convergency depends on the element discretizations of the domain in case of uncertainties too.

- Further, the concept of fuzzy theory has been hybridized with randomness. As such, interval/fuzzy numbers have been introduced in the numerical techniques viz. Euler Maruyama and Milstein methods to handle fuzzy stochastic differential equations. Accordingly, interval/fuzzy Euler Maruyama and Milstein methods have been proposed. The developed methods are then used to solve various problems viz. Black-Scholes, Langevin and uncertain point kinetic neutron stochastic differential equations. Here, the uncertain variation of neutron populations are analysed by considering two random samples. Obtained results in special case of stochastic are compared with the existing ones.

- Finally, fuzzy theory has been combined with the wavelet method. The proposed interval/fuzzy arithmetic (Chapter 3) has been incorporated in wavelet theory and interval/fuzzy wavelet method is proposed. An ODE with interval/fuzzy coefficient has been considered for the demonstration of the developed Interval/Fuzzy Wavelet Method (I/FWM). Obtained uncertain results are compared with known results in special cases.

Although we have studied the above problems in detail and in a systematic way but we do not claim that the proposed methods are most efficient and best. As such, there are few limitations on the proposed methods and we may identify various scopes of improvement. Accordingly these limitations and scopes may open new vistas for future research which are discussed next.

10.2 Future Directions

It is already mentioned above that we may not claim the proposed methods are most general and full proof for solving any type of fuzzy system of linear equations, fuzzy eigenvalue problems and govering fuzzy differential equations. As such, there exist few gaps which may be identified as the future direction of research and those are incorporated below.

- Hybrid type of fuzzy numbers may be introduced in the fuzziness viz. by taking combinations of different fuzzy numbers in the coefficients, variables and initial/boundary conditions.

- Theoretical concepts regarding the methods for the solution of fuzzy and fully fuzzy system of linear equations may be investigated in greater details.

- Similarly, theoretical concepts regarding the numerical methods for the solution of eigenvalue problems may also be investigated.

- Numerical methods along with existence and uniqueness etc. may also be studied for the solution of fuzzy and fully fuzzy system of linear equations.

- Present analysis of uncertain steady and unsteady state heat transfer problems may further be extended to complicated domains.

- Uncertain unalysis of various core neutron diffusion and reactor problems can also be investigated in detail.

- As we know that multiplication and division enhances the interval width. So, it may be a good challenge to develop methods to handle these difficulties.

- Although, we have hybrididsed the fuzzy theory and stochastic, but other challenges may be to define these in general form.

- Another difficulty has been in the subtraction of identical interval/fuzzy numbers. As such one should develop techniques to handle these difficulties. On the other hand, one may easily solve interval/fuzzy system of equation $\tilde{A}\tilde{X} = \tilde{b}$ if we develop methods to find $\tilde{A}^{-1}$.

- Finally, one may study various other diffusion problems using interval/fuzzy uncertainties.

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List of Publications

(a) Journals (Published):

1. Chakraverty S. and Nayak S., 2013. Non probabilistic solution of uncertain neutron diffusion equation for imprecisely defined homogeneous bare reactor, ***Annals of Nuclear Energy***, 62, pp. 251–259*;*
2. Nayak S. and Chakraverty S., 2013. Non-probabilistic approach to investigate uncertain conjugate heat transfer in an imprecisely defined plate, ***International Journal of Heat and Mass Transfer***, 67, pp. 445–454*;*
3. Nayak S. and Chakraverty S., 2015. Fuzzy finite element analysis of multigroup neutron diffusion equation with imprecise parameters, ***International Journal of Nuclear Energy Science and Technology***, 9, pp. 1-22*;*
4. Nayak S., Chakraverty S. and Datta D., 2014. Uncertain spectrum of temperatures in a non-homogeneous fin under imprecisely defined conduction-convection system, ***Journal of Uncertain Systems***, 8 (2), pp. 123-135*;*
5. Nayak S. and Chakraverty S., 2013. A new approach to solve fuzzy system of linear equations, ***Journal of mathematics and computer science***, 7, pp. 205-212;
6. Chakraverty S. and Nayak S., 2012. Fuzzy finite element method for solving uncertain heat conduction problems, ***Coupled Systems Mechanics***, 1 (4), pp. 345-360;
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(b) Journals (Communicated):

1. Nayak S. and Chakraverty S., Numerical solution of fuzzy eigenvalue problems for uncertain systems (*Control and Cybernetics*);
2. Nayak S. and Chakraverty S., Numerical solution of moving plate problem with imprecisely defined parameters (*Computer Modeling in Engineering and Sciences*);
3. Nayak S. and Chakraverty S., Numerical solution of fuzzy stochastic differential equation (*Journal of Intelligent & Fuzzy Systems*);
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5. Nayak S. and Chakraverty S., Numerical solution of Langevin stochastic differential equation with uncertain parameters (*To be submitted*).

(c) Conferences (Published):

1. Nayak S. and Chakraverty S., 2014. Numerical Solution of Two Group Uncertain Neutron Diffusion Equation for Multi Region Reactor, *Advances in Control and Optimization of Dynamical Systems*, Indian Institute of Technology, Kanpur, 3 (1);
2. Nayak S. and Chakraverty S., 2013. Fuzzy finite element approach to solve uncertain neutron diffusion equation for imprecisely defined homogeneous rectangular bare reactor, *Third International conference of Gwalior Academy of Mathematical Sciences (GAMS) and Third IFIP International conference on Bioinformatics,* Maulana Azad National Institute of Technology, Bhopal;
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(d) Book chapters:

1. Chakraverty S. and Nayak S., 2013. Fuzzy Finite Element Method in Diffusion Problems, Mathematics of Uncertainty Modeling in the Analysis of Engineering and Science Problems, *IGI Publication*, USA, edited by S. Chakraverty;
2. Nayak S. and Chakraverty S., 2015. Interval Wavelet Method in Solving Diffusion Equations, Generalized and Hybrid Set Structures and Applications for Soft Computing, *IGI Publication*, USA, edited by Sunil Jacob John, (Accepted).